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The Laminar Boundary Layer in Compressible Flow

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THE LAMINAR BOUNDARY LAYER IN COMPRESSIBLE FLOW

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[Plate 1]

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The paper is concerned with the integration of the laminar boundary-layer equations for a compressible fluid and is in three parts.

In Part I the boundary-layer equations for a compressible fluid are derived, reduced to non-dimensional form, and their relation to the corresponding equations for an incompressible fluid discussed. Methods of integrating them are considered, and it is shown that, provided there is no pressure gradient in the main stream, the methods employed for incompressible flow are of practical value. If there is a pressure gradient, then the complications introduced by compressibility are such that general algebra must cease and numerical integration take its place at an early stage. This means that approximate methods (such as Pohlhausen's) of calculating separation lose their simplicity, and there are indications that their accuracy will also suffer; so it is natural to consider the practicability of direct integration of the equations, probably by series expansions.

In Part II, suitable expansions in one independent variable with coefficients which are functions of the other are obtained. It is found that the independent variables can be so chosen that the differential equations for the coefficients in the expansions have the same general structure as for an incompressible fluid. The boundary conditions and the limiting forms of the equations for zero Mach number are investigated. The application of iterative methods to the equations is discussed.

In Part III the ENIAC is briefly described, and the methods of applying it to obtain solutions of the equations derived in Part II are described in some detail. It is shown that, by proper choice of independent variable, the results for zero pressure gradient can be put into a form in which they vary only slowly with $\mathscr{M}_1^2$. Linear interpolation in $\mathscr{M}_1^2$ between the tabulated values will thus provide reliable first estimates of these quantities, and the accuracy can be improved, if required, by an iterative process.

Tables of results are given.

1. Introduction

This paper is concerned with integrating the laminar boundary-layer equations for a compressible fluid; the corresponding problem for an incompressible fluid is treated in *Modern developments in fluid dynamics* (Goldstein 1938), Chapters III, IV and XIV, and, to avoid an extensive bibliography, references (in the form F.D. p. ...) will be confined to this work so far as possible, and familiarity with its contents must be assumed.

The study of the flow of a compressible fluid, or gas dynamics, brings difficulties of notation. The subject involves hydrodynamics (or fluid dynamics), thermodynamics, and in its practical applications aeronautics and steam engineering. These subjects have competing notations and therefore a compromise is necessary. The solution adopted here is to follow the notation of F.D. so far as possible, but, to avoid confusion with the gas constant (R), $\mathscr{R}$ is used for the Reynolds number and for logical consistency $\mathscr{M}$ for the Mach number; E and I are used for the internal energy and enthalpy or 'total heat' respectively.

Historically the work to be described arose first from an attempt to estimate, if only qualitatively, the effect of the boundary layer on the aerodynamic force coefficients of a projectile, a study which emphasized the great dearth of information and led to the development of certain approximate methods; and secondly from an attempt to calculate the position of separation of the boundary layer. The photographs of figure 1, plate 1, are included as a matter of interest to show the kind of conditions under which separation occurs in external ballistics. The values of $\mathscr{R}$ involved range from about $\frac{1}{4}$ million in a small supersonic wind tunnel up to 100 million or more for a large shell, and it seems likely that in covering this range laminar and turbulent boundary layers are involved, but there are reasons for thinking that in applications to supersonic wind tunnels the boundary layer of the model is more likely to be laminar over most of the range. Therefore while both cases are likely to be, ultimately, of practical importance, the obvious course is to concentrate first on the laminar layer which seems to be of immediate value in model work. An additional reason is provided by the fact that the laminar layer is much more tractable analytically.

PART I. GENERAL SURVEY AND DERIVATION OF THE EQUATIONS

2. Preliminary considerations

It is well known that, although, strictly speaking, the influence of viscosity extends over the whole field influenced by the body, solutions of the boundary-layer equations possess the property of approximating closely to their asymptotic values a short distance from it, and that thereafter, for all practical purposes, the influence of viscosity can be neglected. It is therefore convenient to regard the boundary layer as increasing the thickness of the body by the displacement thickness and to take the velocity of slip of the inviscid solution for the body thus modified to be the velocity at the outer edge of the boundary layer. This point is explicitly made because with compressible flow and supersonic velocities the influence of the body is confined only to a portion of the fluid, and the velocity changes in that portion may be much greater. Therefore the distinction between the 'free stream', which is the inviscid flow before it is influenced by the body and is analogous to the 'conditions at infinity' in incompressible flow, and the 'main stream', which is the flow after it has been influenced by the body, is important. A subscript 1 will be used to denote the 'free stream' value of any quantity, p_1 denoting the static pressure in the free stream; a subscript M to denote the 'main stream' value when it is not necessary to particularize the exact position; a subscript s to denote that the 'main stream' at the outer edge of the boundary layer is involved, U_s denoting the velocity (not necessarily constant) at the outer edge of the boundary layer; and subscript w to denote surface of the body, T_w denoting the temperature of the fluid in contact with the body.

The displacement and momentum thicknesses for a compressible fluid are so defined as to leave their physical interpretations the same as for an incompressible fluid; thus

$$\delta_c^* \equiv \int_0^M \left(1 - \frac{\rho u}{\rho_M U_M}\right) dy, \tag{2·1}$$

$$\vartheta_c \equiv \int_0^M \frac{\rho u}{\rho_M U_M}\left(1 - \frac{u}{U_M}\right) dy. \tag{2·2}$$

The convention about non-dimensional coefficients is to take for the denominator the dynamic head in the free stream, for instance:

$$c_f \equiv \tau_0 / \tfrac{1}{2}\rho_1 U_1^2, \quad \text{the local skin friction coefficient,} \tag{2·3}$$

$$C_f \equiv \frac{1}{x}\int_0^x \tau_0\, dx / \tfrac{1}{2}\rho_1 U_1^2, \quad \text{the average skin friction coefficient from 0 to } x. \tag{2·4}$$

In the study of the boundary-layer equations of an incompressible fluid it is customary to concentrate on solutions of the simplest case that can be taken, namely, the flat plate at zero incidence with or without a pressure gradient in the main stream. There are two reasons for this; first, the analytical and numerical complications are minimized; and secondly, the opinion has been expressed that 'for most practical purposes it would seem to be sufficiently accurate to use the skin friction curves for the flat plate to predict the skin friction drag of a streamline body' (F.D. p. 515). In the present case the complications are greater, and skin friction in all the cases studied up to now has constituted a smaller percentage of the total drag so that there is every reason for concentrating on the flat-plate case. For the sake of completeness and to make comparison between the compressible and incompressible cases easy, the boundary-layer equations for two-dimensional flow along a curved surface and for axisymmetric flow past a rotating solid of revolution are derived later and are found not to differ essentially from their counterparts in the case of an incompressible fluid; but it seems unlikely that the enhanced accuracy of their solutions would compensate—save in an exceptional case—for the greater labour involved. Finally, in applying flat-plate calculations to actual bodies it must be remembered that, with zero pressure gradient, $U_s = U_1$, *ex hypothesi*, and so care must be taken to relate the free-stream and main-stream velocities correctly.

3. Derivation of the boundary-layer equations

It is convenient to begin from the general equations of motion of a fluid all of whose physical properties vary; in vector notation these equations are (Crocco 1939; Emmons & Brainerd 1941; Cope 1942):

The equation of continuity or conservation of mass:

$$\frac{d\rho}{dt}+\rho\Delta = 0. \tag{3·1}$$

The Navier-Stokes equation or conservation of momentum:

$$\rho\frac{d\bar{q}}{dt} = \rho\bar{F}-\nabla p+\mu\nabla^2\bar{q}+\tfrac{1}{3}\mu\nabla\Delta+2\{(\nabla\mu)\ \nabla\}\bar{q}+(\nabla\mu)\ \bar{\zeta}-\tfrac{2}{3}\Delta(\nabla\mu). \tag{3·2}$$

The energy equation or conservation of energy:

$$\rho\frac{dE}{dt}+p\Delta = \Phi+\nabla\{(K\nabla)\ T\}. \tag{3·3}$$

In none of the references cited above is this equation given in its most general terms. Careful examination of its derivation shows that (3·3) is restricted neither to a perfect gas nor to one whose specific heats are constant, but is applicable to any fluid provided E is a function of T only. The dissipation function is given by

$$\Phi = \mu[2\nabla\{(\bar{q}\nabla)\ \bar{q}\}+\bar{\zeta}^2-2\bar{q}\nabla\Delta-\tfrac{2}{3}\Delta^2], \tag{3·4}$$

and is unaffected in form by the fact that viscosity is a variable.

An alternative form of the energy equation can be obtained in terms of the enthalpy; it is

$$\rho\frac{dI}{dt}-\frac{dp}{dt} = \Phi+\nabla\{(K\nabla)\ T\}, \tag{3·5}$$

but since in this form it is now assumed that both E and I are functions of T only this equation is necessarily restricted to a perfect gas, but no restriction to constant specific heats is as yet implied.

There are two main forms of the boundary-layer equations, namely, those for a compressible and those for an incompressible fluid. But in many applications of the latter it has been customary to ignore the energy equation, and to solve the mass and momentum equations with only velocity as the dependent variable, and it has been found that this procedure is adequate for the purpose in view. These applications are the subject of F.D. Chapters III and IV. But sometimes, for instance if heat sources are present, a knowledge of the temperature is needed, and the existence of a 'temperature (boundary) layer' as well as a 'velocity (boundary) layer' must be recognized and the energy equation used. These applications, which are the subject of F.D. Chapter XIV, require an apparently illogical system of equations in which the energy equation has been added to the incompressible forms of the mass and momentum equations, and in which, (i) the velocity of sound is infinite in spite of the fact that (small) variations of pressure density and temperature are permitted and (ii) the physical properties of the fluid (such as viscosity) are constant. The underlying rationale of this system and its justification is discussed with the aid of the equations in reduced non-dimensional form in § 7.

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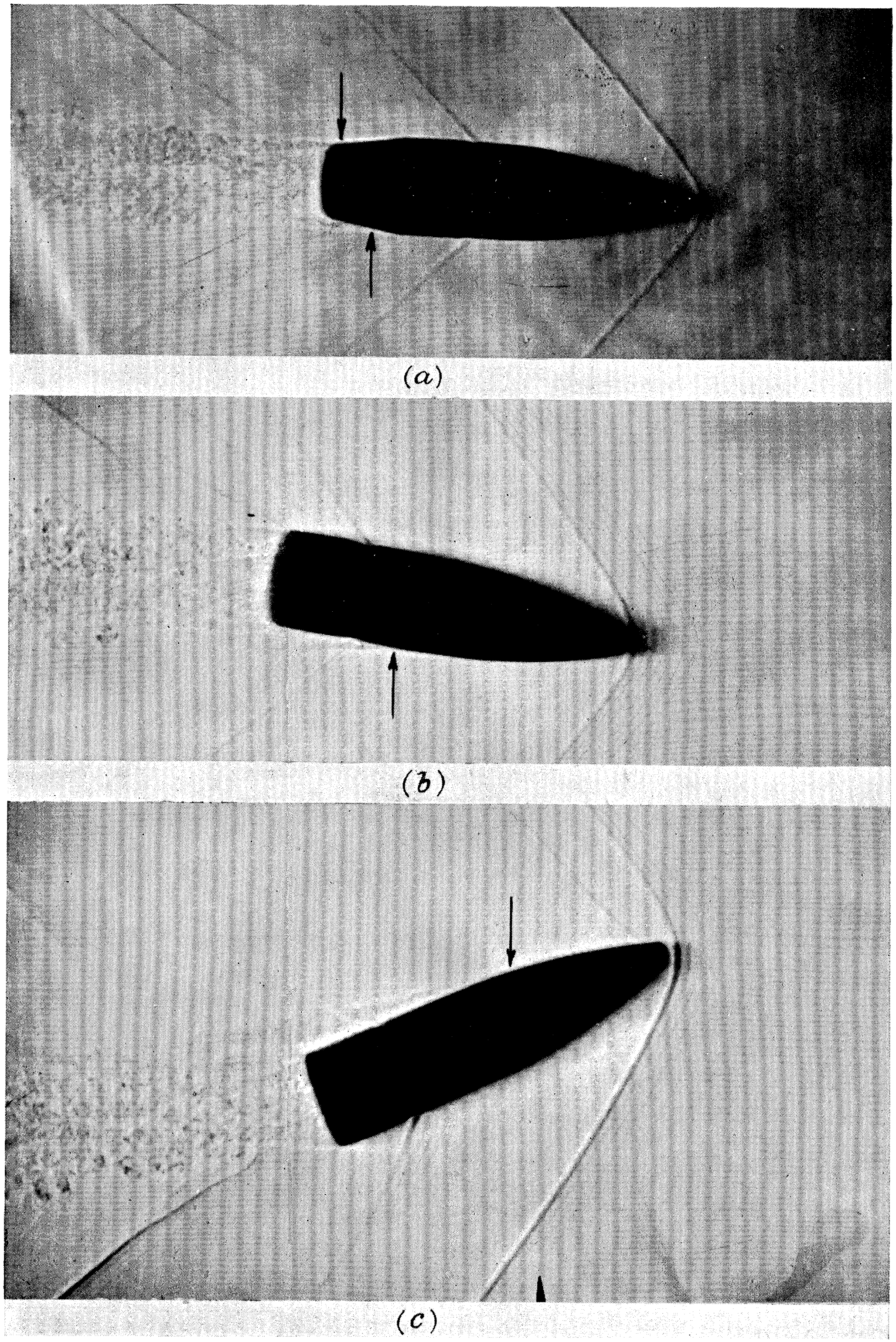

FIGURE 1. Photographs of 0·303 in. bullets in flight showing boundary layer separation. Nominal velocity 1700 ft. per sec. Mach number about $1\frac{1}{2}$. Arrows point to approximate position of separation. (*a*) Mark VIII z. Yaw small. (*b*) Mark VII. Yaw 10°. (*c*) Mark VII. Yaw 25°.

(*Facing p.* 4)

In the following discussion the boundary-layer equations are derived for a compressible fluid whose physical properties (viscosity, thermal conductivity, etc.) are functions of temperature, and without restriction on the Mach number and temperature differences existing in the field of flow. The applications are restricted to perfect gases for which the Prandtl number ($\sigma \equiv \mu C_p/k$) is of order unity, which implies that the velocity and temperature layers are of the same order of thickness, and are confined to steady motion; it is also assumed that there are no sources of heat in the body or fluid. The derivation follows the lines of F.D. pp. 610 to 615, and will be confined to modifications of that treatment which are necessary because the restrictions that

$$\frac{1}{p}\frac{dp}{dt}, \quad \frac{1}{T}\frac{dT}{dt} \quad \text{and} \quad U_M^2/C_p(T_M-T_w)$$

are small have been removed, and because the physical properties are no longer regarded as constants. As a result, some terms in the full equations can no longer be neglected and will be designated 'new'.

Take then from any convenient origin, usually the leading edge in the case of a flat plate, the x-axis along and the y-axis perpendicular to the plate. Then equation (3·1) becomes

$$\frac{\partial}{\partial x}(\rho\mu)+\frac{\partial}{\partial y}(\rho v)=0, \tag{3·6}$$

and since neither $\frac{1}{p}\frac{dp}{dt}$ nor $\frac{1}{T}\frac{dT}{dt}$ can be regarded as small, nor can $\frac{1}{\rho}\frac{d\rho}{dT}$, and the equation cannot be reduced even approximately to the incompressible form.

Equation (3·2) yields two equations, namely,

$$\left(\rho u\frac{\partial}{\partial x}+\rho v\frac{\partial}{\partial y}\right)u=-\frac{\partial p}{\partial x}+\frac{4}{3}\left(\mu\frac{\partial^2 u}{\partial x^2}+\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial y^2}+\frac{\partial\mu}{\partial y}\frac{\partial u}{\partial y}+\tfrac{1}{3}\mu\frac{\partial^2 v}{\partial x\partial y}-\frac{2}{3}\frac{\partial\mu}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial x}, \tag{3·7a}$$

$$\left(\rho u\frac{\partial}{\partial x}+\rho v\frac{\partial}{\partial y}\right)v=-\frac{\partial p}{\partial y}+\frac{4}{3}\left(\mu\frac{\partial^2 v}{\partial y^2}+\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial y}\right)+\mu\frac{\partial^2 v}{\partial x^2}+\frac{\partial\mu}{\partial x}\frac{\partial v}{\partial x}+\tfrac{1}{3}\mu\frac{\partial^2 u}{\partial x\partial y}-\frac{2}{3}\frac{\partial\mu}{\partial y}\frac{\partial u}{\partial x}+\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial y}. \tag{3·7b}$$

In (3·7a) the only 'new' term which must be considered is $\frac{\partial\mu}{\partial y}\frac{\partial u}{\partial y}$, so the equation becomes

$$\left(\rho u\frac{\partial}{\partial x}+\rho v\frac{\partial}{\partial y}\right)u=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{3·8a}$$

In (3·7b) all the 'new' terms are $O(\delta)$ or less and can be neglected, the equation becoming

$$0=-\frac{\partial p}{\partial y}. \tag{3·8b}$$

Equation (3·3) with Φ written out in full becomes

$$\rho C_v\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)T+p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$=k\frac{\partial^2 T}{\partial x^2}+\frac{\partial k}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial^2 T}{\partial y^2}+\frac{\partial k}{\partial y}\frac{\partial T}{\partial y}-\tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+2\mu\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\mu\left\{\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}^2, \tag{3·9}$$

and the following terms must be considered: all the left-hand side, and the terms

$$k\frac{\partial^2 T}{\partial y^2},\quad \frac{\partial k}{\partial y}\frac{\partial T}{\partial y}\quad \text{and}\quad \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

So the equation becomes

$$\rho C_v\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{3·10}$$

and is not yet restricted to a perfect gas; but with a view to further developments it is more convenient to have this equation in another form. First, for a perfect gas $p = R\rho T$ and $C_p = \gamma C_v = \gamma R/(\gamma-1)$, so the equation can be written (now restricted to a perfect gas, but not to constant specific heats)

$$\frac{\gamma}{\gamma-1}p\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{1}{\rho}\right)+\frac{u}{\gamma-1}\frac{\partial p}{\partial x}=\frac{p}{R}\frac{\partial}{\partial y}\left\{k\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right\}+\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{3·11}$$

and finally, since $\sigma = \gamma R\mu/(\gamma-1)\,k$, and with the further restriction to constant specific heats,

$$\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{1}{\rho}\right)+\frac{1}{\gamma}\frac{u}{p}\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left\{\frac{\mu}{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right\}+\frac{\gamma-1}{\gamma}\frac{\mu}{p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3·12}$$

which is the form required.

Collecting results we have the equations of motion for the boundary layer of a flat plate with a (possibly) supersonic main stream in what will be regarded as their standard form. They are:

The equation of continuity:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0. \qquad (a)$$

The equation of momentum:

$$\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \qquad (b)$$

with

$$\frac{\partial p}{\partial y}=0.$$

The equation of energy:

$$\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{1}{\rho}\right)+\frac{1}{\gamma}\frac{u}{p}\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left\{\frac{\mu}{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right\}+\frac{\gamma-1}{\gamma}\frac{\mu}{p}\left(\frac{\partial u}{\partial y}\right)^2. \qquad (c)$$

(3·13)

For the case of two-dimensional flow near a cylindrical body take the origin at the forward stagnation point, the axis of x along and the axis of y perpendicular to the surface. A discussion, which need not be given, on identical lines leads to a system of equations which differ from the system (3·13) only in having

$$\frac{\partial p}{\partial y}=-\rho K u^2[=O(1)], \tag{3·14 b}$$

instead of $\partial p/\partial y = 0$, where K is the curvature of the body.

For the case of axisymmetric flow past a rotating solid of revolution take the same origin and axes as above and (following F.D. p. 128) let r be the distance from the axis. Also let

w be the peripheral speed with an upper bound of the same order as U_M. Again the discussion need not be given; the resulting system of equations is

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial y}(\rho r v)=0, && (a)\\
&\left.\begin{aligned}
&\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u-\rho\frac{w^2}{r}\frac{\partial r}{\partial x}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\\
&-\rho\frac{w^2}{r}\frac{\partial r}{\partial y}=-\frac{\partial p}{\partial y}[=O(1)],\\
&\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)w=\mu\frac{\partial^2 w}{\partial y^2}-\frac{\partial \mu}{\partial y}\frac{\partial w}{\partial y},
\end{aligned}\right\} && (b)\\
&\rho\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{1}{\rho}\right)+\frac{1}{\gamma}\frac{u}{p}\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left\{\frac{\mu}{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right\}+\frac{\gamma-1}{\gamma}\frac{\mu}{p}\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2. && (c)
\end{aligned}\right\} \quad (3\cdot15)$$

As stated earlier, these equations are given for the sake of completeness, and to demonstrate their essential parallelism to the incompressible system; but they will not be used in the sequel which is confined to the flat plate at zero incidence.

The flat-plate equations as a system seem to have been first given explicitly by Busemann (1935), but it should be noted that Pohlhausen (F.D. p. 627) successfully integrated the energy equation of the incompressible system for zero pressure gradient many years before. Later work has shown that some of his conclusions, and notably that $I+\frac{1}{2}\sqrt{\sigma}\,(u^2+v^2)=$ const. is a good approximation to the accurate solution of the energy equation, are surprisingly little affected by the change to a more complicated system.

At the boundary of the plate $u=v=0$, and we get

$$\begin{aligned}
\frac{\partial p}{\partial x}&=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)_w\\
&=\mu_w\left(\frac{\partial^2 u}{\partial y^2}\right)_w+\left(\frac{\partial \mu}{\partial y}\frac{\partial u}{\partial y}\right)_w\\
&=\mu_w\left(\frac{\partial^2 u}{\partial y^2}\right)_w+\left(\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\frac{\partial u}{\partial y}\right)_w \quad \text{at a given } x,
\end{aligned}$$

and since the only case being considered is that of no heat sources in either gas or plate, there is, in the steady state, no heat transfer between them, so

$$\left(\frac{\partial T}{\partial y}\right)_w=0 \quad \text{all } x,$$

and the boundary condition gives
$$\frac{\partial p}{\partial x}=\mu_w\left(\frac{\partial^2 u}{\partial y^2}\right)_w. \quad (3\cdot16)$$

In the main stream the velocity is related to the pressure gradient by

$$\frac{\partial p}{\partial x}=\rho_M U_M\frac{dU_M}{\partial x}. \quad (3\cdot17)$$

The momentum-integral equation (often loosely called the momentum equation), is obtained by integrating (3·13b) through the boundary layer and using (3·13a) to eliminate v. It is

$$\frac{d\vartheta_c}{dx}+(\delta_0^*+2\vartheta_c)\frac{1}{U_M}\frac{dU_M}{dx}+\vartheta_c\frac{1}{\rho_M}\frac{d\rho_M}{dx}=\tfrac{1}{2}c_f.$$

In the main stream $I+\frac{1}{2}U_m^2$ is constant and the adiabatic law holds; these facts can be used to eliminate the term in $d\rho_M/dx$ from this equation and give the standard form

$$\frac{d\vartheta_c}{dx}+\left\{\delta_0^*+(2-\mathscr{M}_M^2)\,\vartheta_c\right\}\frac{1}{U_M}\frac{dU_M}{dx}=\tfrac{1}{2}c_f. \tag{3·18}$$

Finally, the same method could be applied to the energy equation on the lines of F.D. pp. 614 and 615, to yield an energy-integral equation, but it does not seem that any useful purpose would be served by the equation thus obtained since an approximate solution of the energy equation is known.

4. Methods of solution

Obviously any method of solution, if it is to be of practical use, must be capable of yielding numerical results and therefore all methods of solution are in a sense numerical. Nevertheless, it is convenient to divide methods of solution into categories which are distinguished by the amount and nature of the preliminary algebraic treatment. From this point of view the several categories are:

(*a*) Methods depending on the momentum equation, for instance, the Pohlhausen (F.D. p. 159) method of calculating separation.

(*b*) Methods depending on a change of independent variable, for instance von Mises's (F.D. p. 127) transformation of the equations with x and ψ as independent variables as used (F.D. p. 164) for the form of 'inner' and 'outer' solutions to calculate separation.

(*c*) Methods depending on a series solution of the equations with or without a preliminary transformation, for instance, the original and now classical solution (F.D. p. 135) for the special case of zero pressure gradient.

(*d*) Methods which integrate the equations numerically as they stand, for instance, by replacing derivatives by finite differences and solving the resulting equations as such.

Since this section is primarily concerned with algebra, consideration of 4 (*d*) will be postponed until later in the paper when numerical methods in general are under discussion.

(*a*) *Momentum-integral equation*

The general principles of methods based on the momentum-integral equation for incompressible flow are given in F.D. pp. 131 to 134 and 156 to 160. They will be followed here, and in addition it is analytically convenient to define a quantity α, a function of $\mathscr{M}$ and related to it by

$$(1-\alpha)\left(1+\frac{\gamma-1}{2}\mathscr{M}^2\right)=1$$

or

$$\alpha=\frac{\gamma-1}{2}\mathscr{M}^2\bigg/\left(1+\frac{\gamma-1}{2}\mathscr{M}^2\right). \tag{4·1}$$

α as a function of $\mathscr{M}$ is plotted as figure 2.

Under these circumstances

$$\delta_c^*=\delta\int_0^1\left(1-\frac{(1-\alpha)f}{1-\alpha f^2}\right)d\eta,\quad \vartheta_c=\delta\int_0^1\frac{(1-\alpha)f\,(1-f)}{1-\alpha f^2}\,d\eta,\quad \tau_0=\frac{\mu_w U_s}{\delta}f'(0),$$

and the relations (3·16, 3·17) give

$$\rho_s \delta \frac{dU_s}{dx} = \mu_w \Lambda_c, \quad \Lambda_c = -f''(0).$$

The integrals can be reduced to

$$\delta_c^* = \delta\left\{1 - \frac{1-\alpha}{2\sqrt{\alpha}}(J_1 - J_2)\right\}, \quad \vartheta_c = \delta\left\{\frac{1-\alpha}{\alpha}\left(1 - \frac{1-\sqrt{\alpha}}{2}J_1 - \frac{1+\sqrt{\alpha}}{2}J_2\right)\right\},$$

where J_1 and J_2 are defined by

$$J_1 \equiv \int_0^1 \frac{d\eta}{1-\sqrt{\alpha f}}, \quad J_2 \equiv \int_0^1 \frac{d\eta}{1+\sqrt{\alpha f}}.$$

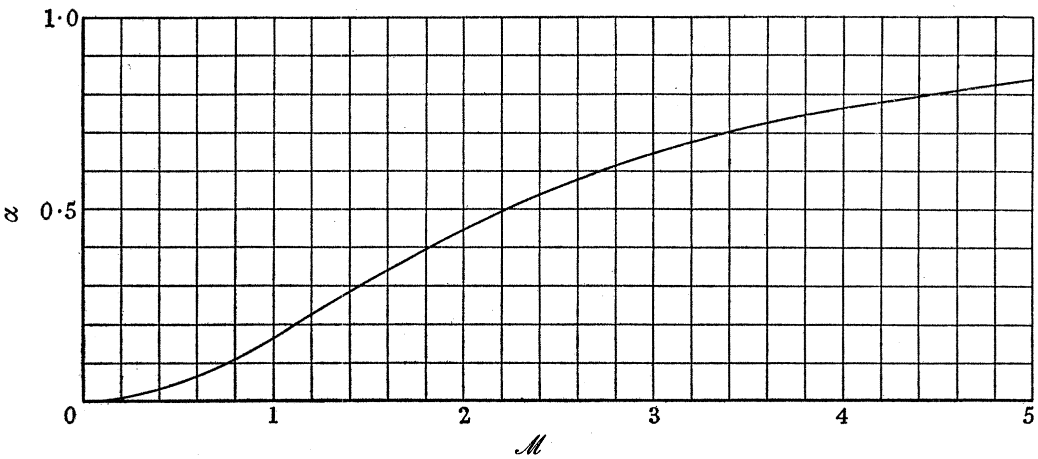

FIGURE 2. α as defined by $(1-\alpha)\left(1+\frac{\gamma-1}{2}\mathscr{M}^2\right) = 1$.

It is also assumed that the generalized Bernoulli equation

$$\text{total energy} = I + \tfrac{1}{2}(u^2+v^2) = \text{const.}$$

is a solution of the energy equation (3·13*c*). This is justified on the grounds that (i) it is assumed in deriving the original equations that $\sqrt{\sigma} = O(1)$, (ii) it can be shown that for $\sigma = 1$, $I + \frac{1}{2}(u^2+v^2) = \text{const.}$ is an exact solution in certain circumstances (Cope 1942), (iii) numerical solutions extant show that $I + \frac{1}{2}\sqrt{\sigma}\,(u^2+v^2) = \text{const.}$ is an excellent approximation, over the whole range so far covered, to the true solution. The successful application of the method, therefore, depends only on finding forms for f which satisfy the conditions previously laid down, but it seems likely that the choice will be much more restricted than for incompressible flow.

For zero pressure gradient dU_s/dx, and with it Λ_c vanishes and (3·18) becomes

$$\frac{d\vartheta_c}{dx} = \tfrac{1}{2}c_f. \tag{4·2}$$

If we now assume the trigonometric form

$$f = \sin\frac{\pi}{2}\eta \tag{4·3}$$

(F.D. p. 157), J_1 and J_2 can be evaluated comparatively simply in finite terms to give

$$\left.\begin{aligned}
\delta_c^*/\delta &= 1-\frac{2}{\pi}\sqrt{\frac{1-\alpha}{\alpha}}\arctan\sqrt{\frac{\alpha}{1-\alpha}},\\
J_\vartheta\equiv\vartheta_c/\delta &= \frac{1-\alpha}{\alpha}\left\{\frac{2}{\pi}\sqrt{\frac{\alpha}{1-\alpha}}\arctan\sqrt{\frac{\alpha}{1-\alpha}}-\frac{1-\sqrt{(1-\alpha)}}{\sqrt{(1-\alpha)}}\right\},\\
\frac{\delta}{x}\sqrt{\mathscr{R}_x} &= \sqrt{\pi}\,\{(1-\alpha)^\beta J_\vartheta\}^{-\frac{1}{2}},\\
c_f\sqrt{\mathscr{R}} &= \sqrt{\pi}\sqrt{[J_\vartheta/(1-\alpha)^\beta]},\\
C_f\sqrt{\mathscr{R}} &= 2\sqrt{\pi}\sqrt{[J_\vartheta/(1-\alpha)^\beta]},
\end{aligned}\right\} \tag{4·4}$$

where $J_\vartheta\equiv\vartheta_c/\delta$, a function of α, that is, of $\mathscr{M}$ only, and it is assumed that $\mu\propto T^\beta$.

This solution has been compared with an exact solution (Brainerd & Emmons 1942) and seems to be in reasonable agreement. For the same value of β and over the range $0<\mathscr{M}\leqslant\sqrt{10}$ the exact solution shows that $C_f\sqrt{\mathscr{R}}$ decreases from 1·327 to 1·209 (about 10 %), whereas the trigonometric form (4·3) shows an increase from 1·316 to 1·367 (about 3 %). It is probable that this change in $C_f\sqrt{\mathscr{R}}$ is for all practical purposes negligible, and therefore the sine solution has its uses as a convenient means of obtaining approximate values quickly. Figures 3 and 4 have been included to give an idea of the sort of variations of δ^* and ϑ_c which occur as $\mathscr{M}$ increases.

If there is a pressure gradient then (3·18) must be used in full, but before proceeding to the compressible case it is convenient to revert for a moment to incompressible flow and consider the Pohlhausen method of calculating separation for the most general form of velocity distributions subject to the restriction that the functions used must form a one parameter set with parameter Λ. This form is

$$f=F+\Lambda G,$$

where F and G are unspecified functions of η; the momentum equation becomes, in the notation of F.D. p. 159,

$$\frac{dZ}{dx}=h(\Lambda)\frac{d^2U_s}{dx^2}Z^2+g(\Lambda)\frac{1}{U_s}, \tag{4·5}$$

where

$$h(\Lambda)=-\frac{2d-4e\Lambda}{c+3d\Lambda-5e\Lambda^2}$$

$$g(\Lambda)=\frac{2F'(0)+\{2G'(0)-2a-4c\}\Lambda+(2b-4d)\Lambda^2+4e\Lambda^3}{c+3d\Lambda-5e\Lambda^2},$$

and

$$a=1-\int_0^1 Fd\eta,\quad b=\int_0^1 Gd\eta,\quad c=1-a-\int_0^1 F^2d\eta,\quad d=b-2\int_0^1 FGd\eta,\quad e=\int_0^1 G^2d\eta$$

and

$$Z\equiv\frac{\rho\delta^2}{\mu}\quad\text{or}\quad\Lambda\frac{dU_s}{dx}.$$

The fact that $g(\Lambda)$ is of the form (cubic in Λ)/(quadratic in Λ) for several particular forms of F and G has been noted; the point of the above analysis is that it shows that the form of $g(\Lambda)$ as a function of Λ has nothing to do with the form of f as a function of η. A detailed examination of the steps by which (4·5) is derived shows that it arises quite naturally from the assumption that f is a linear function of Λ. The numerator of $g(\Lambda)$ arises from $\vartheta \times dU_s/dx$, and so necessarily involves Λ^3 and the denominator from ϑ and involves Λ^2; thus it can be seen that if f were assumed to be polynomial of the nth degree (an n-tic) in Λ, then $g(\Lambda)$ would be of the form $[(2n+1)$-tic in $\Lambda]/[2n$-tic in $\Lambda]$. The analysis of F.D. pp. 158 to 166 can, of course, be recovered by substituting the quartic form for f in (4·5).

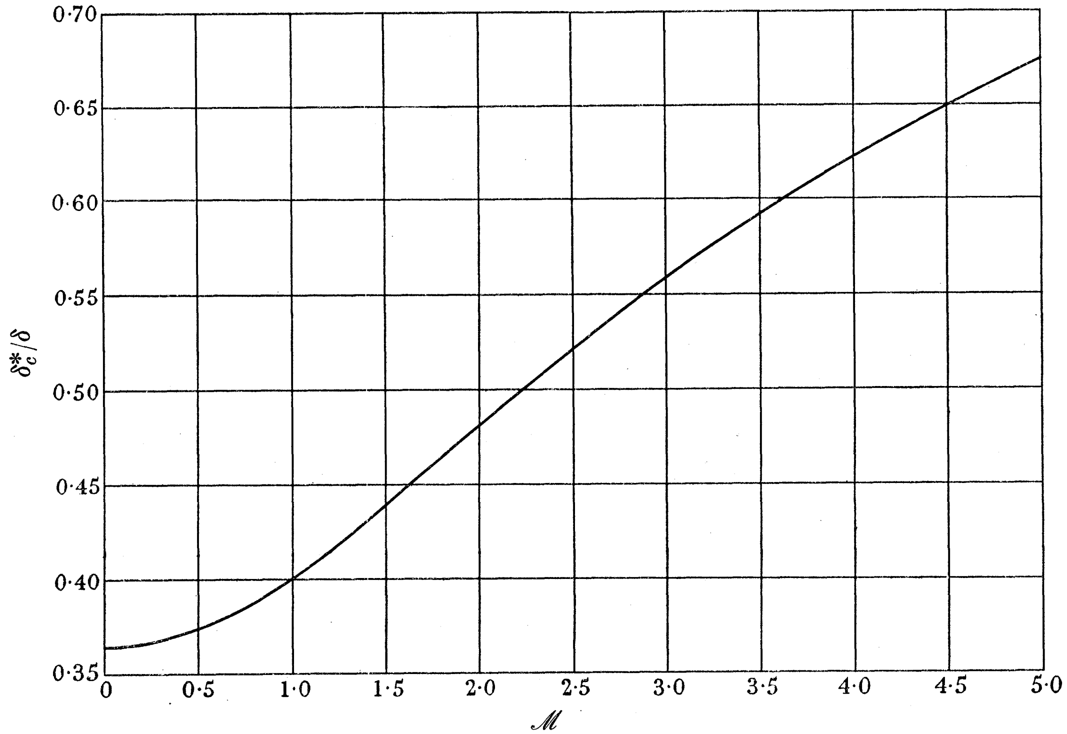

FIGURE 3. Laminar boundary layer, sinusoidal velocity distribution, $\delta_c^* \equiv \int_0^\delta \left(1 - \frac{\rho u}{\rho_1 U_1}\right) dy.$

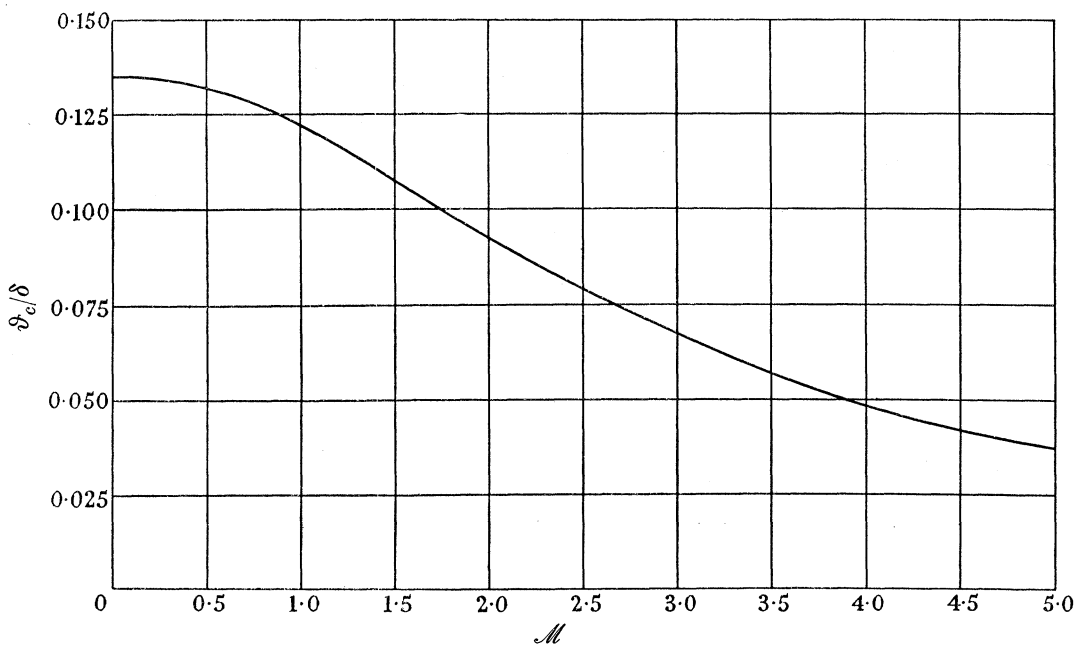

FIGURE 4. Laminar boundary layer, sinusoidal velocity distribution, $\vartheta_c \equiv \int_0^\delta \frac{\rho u}{\rho_1 U_1}\left(1 - \frac{u}{U_1}\right) dy.$

For compressible flow assuming that

$$f = F + \Lambda_c G,$$

(3·18) becomes

$$2\Lambda_c \frac{d}{dx}\left[\delta \frac{1-\alpha_s}{\alpha_s}\left\{1-\frac{J_1+J_2}{2}+\sqrt{\alpha_s}\left(\frac{J_1-J_2}{2}\right)\right\}\right]$$
$$+\frac{\Lambda_c \delta}{\gamma-1}\left[2(\gamma-1)+(\gamma+1)\,\alpha_s+\{(\gamma+1)-\gamma\alpha_s\}\,(J_1-J_2)\right.$$
$$\left.-\{(\gamma-1)-(\gamma+1)\,\alpha_s\}\,(J_1+J_2)\frac{\sqrt{\alpha_s}}{2}\right]\frac{1}{a_s^2}\frac{d\alpha_s}{dx}$$
$$=\{F'(0)+\Lambda_c G'(0)\}\frac{1}{\alpha_s}\frac{d\alpha_s}{dx}. \tag{4·6}$$

In this equation α_s is given as a function of x, J_1 and J_2 as functions both of x (through α_s) and of Λ_c, and as before it is to be regarded as an equation for Λ_c or for one and only one parameter that can be related to it. It should be noted that until J_1 and J_2 have been evaluated it cannot be thrown into the form equivalent to (4·5).

It is immediately obvious that the algebra is going to be considerably heavier, and that difficulties arise because the successful use of the method in general terms requires the algebraic evaluation of integrals of the type $\int_0^1 \frac{d\eta}{1+\Lambda f}$, where $-1<\Lambda<1$ and f involves the dependent variable Λ_c. The possible forms for f are thus drastically restricted to quadratics in η or $\sin\frac{1}{2}\pi\eta$, and even these lead to integrals which change with the numerical value of Λ_c from inverse trigonometric to inverse hyperbolic form and therefore require careful interpretation. The fact that, in any particular case, it might be comparatively simple to evaluate them numerically is here irrelevant, since we are only concerned at the moment with algebraic treatment. Moreover, the restriction to simple forms for f necessarily means that fewer boundary conditions will be satisfied and the accuracy of the solution suffers in consequence. Since Λ is always less than unity it is permissible to expand the denominator of the integrand and integrate term by term, but for high Mach numbers the resulting series is only slowly convergent, and in any case this process is equivalent to assuming that f is a polynomial in Λ_c. The discussion of (4·5) shows that this would lead to the analogues of $h(\Lambda)$ and $g(\Lambda)$ involving high powers of Λ_c, and the discovery of the relevant roots of the latter, an essential preliminary to further developments, might be quite difficult. A further point to be noticed is that even supposing the integrals have been evaluated, one is still confronted with a formidable equation with Λ_c probably an implicit function of the other variables. Finally, attempts to work in terms of $\mathscr{M}_s$ or U_s instead of α_s have not resulted in any essential simplification. The rather obvious inference to be drawn will be given later; the essential point to be noticed at the moment is that the algebraic treatment of the problem has come to a dead stop earlier than is the case with incompressible flow, that no analogue to (4·5) exists, and that numerical methods of solution are necessary if any further progress is to be made.

(*b*) *Change of independent variable*

Equation (3·13*a*) shows that there is a stream function given by

$$\rho_1 \partial\psi/\partial y = \rho u, \quad \rho_1 \partial\psi/\partial x = -\rho v. \tag{4·7}$$

Therefore it is possible to transform the equations into forms (F.D. pp. 126 sqq.) with x and ψ as independent variables. For the particular case of zero pressure gradient (3·13b) becomes (Karman & Tsien 1938)

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial}{\partial \psi^*}\left(u^*\rho^*\mu^*\frac{\partial u^*}{\partial \psi^*}\right),$$

where u^*, x^*, etc., are new non-dimensional variables defined by

$$u^* = u/U_1, \quad x^* = x/L, \quad \rho^* = \rho/\rho_1, \quad \mu^* = \mu/\mu_1 \quad \text{and} \quad \psi^* = (\mathscr{R}^{\frac{1}{2}}/U_1 L)\,\psi,$$

where L is the length of the plate. This equation has been used by Karman & Tsien to obtain values of C_f which agree quite well with exact solutions. They assumed $\sigma = 1$ and used an iterative method. The method of 'inner and outer solutions' (F.D. pp. 164 sqq.) represents an application of the same technique to the problem of calculating separation. No attempt has been made to derive the corresponding equation for compressible flow, since it seems obvious that the algebraic complications would be at least as great as for Pohlhausen's method, and the method has the additional drawback that solution in terms of ψ as independent variable has a very awkward singularity at $\psi = 0$.

(c) *Series solutions*

When there is no pressure gradient the solution of the equations of continuity and momentum for incompressible flow, in terms of a new independent variable proportional to $y/\surd x$, is classical (F.D. pp. 135 to 136); and, as noted earlier, Pohlhausen has given a solution of the energy equation for incompressible flow (F.D. p. 627). If there is a pressure gradient then it is assumed that (3·17) holds in the main stream; to this extent the solution is analytically an approximation, though numerically it will be satisfied to a high degree of accuracy quite close to the plate. For the particular case of a constant velocity gradient

$$U_s = U_1 - U_1' x, \quad \partial p/\partial x = -\rho_1 U_1'(U_1 - U_1' x),$$

and, again for the equations of continuity and momentum only, Howarth (1938 and F.D. p. 172) has obtained a solution in the form

$$2\frac{u}{U_1} \equiv 2u^* = f_0'(\eta) - (8\xi) f_1'(\eta) + (8\xi)^2 f_2'(\eta) - \ldots, \tag{4·8}$$

where
$$\xi = U_1' x/U_1, \quad 2\eta = y(U_1/\nu x)^{\frac{1}{2}} = y(U_1'/\nu\xi)^{\frac{1}{2}}, \tag{4·9}$$

and the boundary conditions are

$$f_j(0) = f_j'(0) = 0 \;\;(\text{all } j) \quad \text{and} \quad f_0'(\infty) = 2, \quad f_1'(\infty) = \tfrac{1}{4}, \quad f_j'(\infty) = 0 \quad (j \geqslant 2).$$

This method of attack has the advantage that by working in terms of $y/\surd x$ and x as independent variables the equations reduce to a system of ordinary equations which can be solved successively. This feature is very important and renders the method practicable.

Emmons & Brainerd (1941, 1942) have shown that for zero pressure gradient the system of equations (3·13) have a solution in which u, $v\surd x$, ρ and T are functions of $y/\surd x$ only, both for constant viscosity and for viscosity a function of temperature. The equations then reduce exactly to a set of ordinary differential equations and solutions which are exact in the numerical sense can be obtained. Solutions for a series of values of $\mathscr{M}$ up to $\surd 10$, with $\beta = 0{\cdot}768$, were obtained, and these, so far as is known, are the only exact solutions extant and have been used as a standard of comparison for any approximate solutions obtained.

5. Discussion

In § 3 the boundary-layer equations for a compressible fluid are derived, and in § 4 the possibility of obtaining solutions in algebraic form is investigated.

If there is no pressure gradient in the main stream and if an approximate solution, tantamount to putting $\sigma = 1$, of the energy equation is used, then any of the methods used for obtaining approximate solutions in the incompressible flow case seem to be practicable. In particular, it is shown that the assumption of the trigonometric form (4·3) for the velocity distribution enables the equations to be integrated comparatively simply in finite terms and the resultant solution is sufficiently close to the 'exact' solution to be acceptable as a convenient approximation.

For the case of a pressure gradient in the main stream the Pohlhausen method has been selected for detailed examination as it is the simplest although the least accurate (F.D. p. 162) of the methods used in the incompressible case. It is shown that with the same approximate solution of the energy equation complications arise from two causes. First, the integrals for displacement and momentum thicknesses involve the velocity distribution and the parameter Λ_c in their denominator. This drastically restricts the range of velocity distributions of practical value and prevents the use of algebraic methods in all but the earliest stages. Secondly, the 'momentum' equation itself is much more complicated and, even if the problem of integration were overcome, would almost certainly involve much heavy algebra to prepare it for the final integration. This, of course, means that algebra must be abandoned at an earlier stage than for incompressible flow and numerical methods employed, and that even then there is no guarantee that the method will be less inaccurate and it has certainly lost its simplicity. Because of this result no attempt has been made to apply the method of 'inner and outer solutions', which again is only approximate, to compressible flow. The conclusion seems to be that any method of calculating the behaviour of the laminar boundary layer in compressible flow when there is a pressure gradient in the main stream is going to involve a major computing operation, and that therefore the proper line of attack is to investigate the practicability of solving the boundary-layer equations (3·13), probably by series expansions on Howarth's lines.

But before numerical results can be obtained it is necessary to adopt a specific law of variation of viscosity with temperature and a value for σ. The important fluid is air which, for all practical purposes, can be regarded as a diatomic perfect gas whose variation of viscosity with temperature is best represented by a formula of Sutherland's type:

$$\mu \propto T^{\frac{3}{2}}/(T+C) \quad \text{with} \quad C \approx 114^\circ\,\text{C}. \tag{5·1}$$

There are advantages, in both analytical and numerical work, in a power law variation

$$\mu \propto T^{\beta}, \tag{5·2}$$

especially if β can be taken to be a simple fraction. The use of a formula of this type as an interpolation formula implies that

$$\beta = 1{\cdot}5 - \frac{T}{T+C}, \tag{5·3}$$

so that β is a function of the temperature range over which the formula is to be used and the extent of the range is important.

Practical applications of numerical work can conveniently be divided into two categories:

(*a*) Measurements in supersonic wind tunnels $90° \leqslant T \leqslant 300°$ K approximately.

(*b*) Measurements in firing trials and other applications in free air $250° \leqslant T \leqslant 600°$ K approximately.

The method of finding β is the usual one of fitting the best straight line to a plot of $\log \mu$ against $\log T$ over the selected range of T, and a certain amount of latitude is permissible, since numerical work (Emmons & Brainerd 1942) shows that the results of calculations are not very sensitive to the choice of β.

Figure 5 is the plot for range (*a*); the slope of the $\log \mu - \log T$ line is 0·884, and for numerical convenience this is rounded up to $\frac{8}{9}$. For range (*b*) the accepted value of β is 0·768, but again for numerical convenience this is rounded down to $\frac{3}{4}$. Accordingly, it is suggested that

for range (*a*) $$90° \leqslant T \leqslant 300° \text{ K}, \quad \mu \propto T^{\frac{8}{9}}, \tag{5·4a}$$

and for range (*b*) $$250° \leqslant T \leqslant 600° \text{ K}, \quad \mu \propto T^{\frac{3}{4}}, \tag{5·4b}$$

are adequate interpolation formulae for the purpose in view.

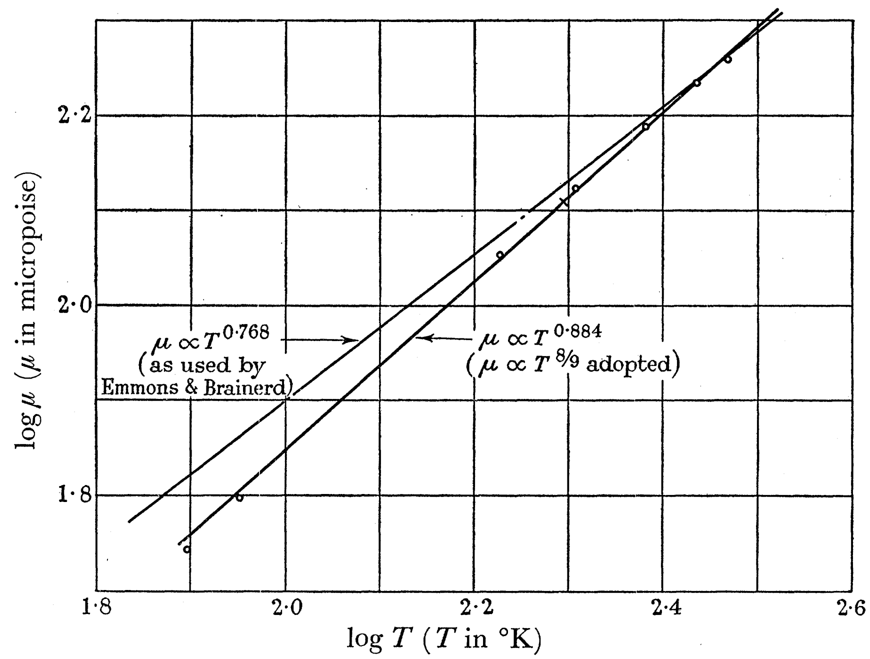

FIGURE 5. Law of variation of viscosity with temperature for air at low temperatures.

The kinetic theory (Jeans 1940, p. 190) gives

$$\sigma = 4\gamma/(9\gamma-5) \quad \text{(a constant)}$$
$$= 0·737 \quad \text{for a diatomic gas with } \gamma = 1·40, \tag{5·5}$$

but an examination of the available data for air (Kaye & Laby 1944) suggests that this value is a few per cent too high, and that the most convenient value which is at the same time consonant with the latest data is

$$\sigma = 0·715 \quad (\surd\sigma = 0·8456, \sqrt[3]{\sigma} = 0·8942). \tag{5·6}$$

It is worth noting in passing, as a curious numerical coincidence, that the simplest (or 'billiard ball') form of the kinetic theory gives

$$\sigma = 1/\gamma = 0·7143\ldots!$$

6. Reduction of the equations to non-dimensional form

It was stated earlier that when there is no pressure gradient the $\partial p/\partial x$ term in (3·13*b*) drops out and that Emmons & Brainerd have shown that the system of equations (3·13) have a solution in which u, $v\sqrt{x}$, ρ and T are functions of $y/\sqrt{x}$ only; the equations can then be reduced exactly to a set of three ordinary differential equations whose solutions can be evaluated mechanically by the differential analyzer. Emmons & Brainerd (1941, 1942) have given solutions of this system for constant viscosity and for various viscosity laws respectively, and these have been used as a standard of comparison for the various approximate solutions given earlier. In addition, their work shows that in this case the boundary layer does not separate from the solid boundary, so that, as in the incompressible case, it is necessary to have retarded flow in order to get separation.

The fact that for no pressure gradient the solution is a function of $y/\sqrt{x}$ only suggests that when there is a pressure gradient it will be possible to work as in the incompressible case in terms of x and $y/\sqrt{x}$ as independent variables, and it can be hoped that the variation of the various quantities with x at constant $y/\sqrt{x}$ will be slow, smooth and easy to handle.

The simplest case to consider seems to be that of constant *pressure gradient* in the main stream, rather than constant *velocity gradient* as taken by Howarth (1938 and F.D. p. 173); moreover measurements show that for flow round a projectile with a streamline base (figure 1, top photograph) the pressure gradient is often nearly constant over the streamlined portion, so that the choice of constant pressure gradient corresponds to an eminently practical case.

For generality let the pressure gradient be written

$$\frac{\partial p}{\partial x} = \rho_1 U_1 U_1' G(U_1' x/U_1), \tag{6·1}$$

where $G(0) = 1$, so that U_1' has the dimensions of a velocity gradient and is positive for retarded flow; for constant pressure gradient $G = 1$. Convenient dimensionless variables are then obtained by substitutions akin to those used by Howarth, namely,

$$\left.\begin{aligned} \xi = U_1' x/U_1, \quad \eta^* = \tfrac{1}{2} y (U_1/\nu_1 x)^{\frac{1}{2}}, \\ u^* = u/U_1, \quad v^* = v(x/U_1 \nu_1)^{\frac{1}{2}} = v(\xi/U_1'\nu_1)^{\frac{1}{2}}. \end{aligned}\right\} \tag{6·2}$$

For an incompressible fluid, η^* as here defined is the same as Howarth's (1938) η. But for a compressible fluid, in order to retain the closest similarity later to Howarth's equations, it is convenient to reserve the symbol η for another quantity which will be introduced later (equation (6·9) below), and which reduces to η^* in the incompressible case. Emmons & Brainerd (1941, 1942) use $\eta = y(U_1/\nu_1 x)^{\frac{1}{2}}$; the factor $\frac{1}{2}$ has been introduced in the definition of η^* to agree with Howarth's usage and to simplify the checking of later formulae by comparison with his.

In terms of these variables (6·1) becomes

$$\frac{\partial}{\partial \xi}\left(\frac{p}{p_1}\right) = \frac{\rho_1 U_1^2}{p_1} G(\xi) = \gamma \mathscr{M}_1^2 G(\xi), \tag{6·3}$$

and (3·13*a*) becomes

$$\left(2\xi \frac{\partial}{\partial \xi} - \eta^* \frac{\partial}{\partial \eta^*}\right) \frac{\rho u^*}{\rho_1} + \frac{\partial}{\partial \eta^*}\left(\frac{\rho v^*}{\rho_1}\right) = 0; \tag{6·4}$$

also

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{U_1'}{2\xi}\left\{2u^*\xi\frac{\partial}{\partial \xi} - (\eta^* u^* - v^*)\frac{\partial}{\partial \eta^*}\right\}. \tag{6·5}$$

Now in equations (3·13 b, c), v only occurs in the combination $\frac{\rho}{\rho_1}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)$, so v^* only occurs in the combination $\frac{\rho}{\rho_1}(\eta^* u^* - v^*)$. It is therefore convenient to write

$$w = \frac{\rho}{\rho_1}(\eta^* u^* - v^*); \qquad (6{\cdot}6)$$

from (6·4) it follows that

$$\frac{\partial w}{\partial \eta^*} = \left(1+2\xi\frac{\partial}{\partial \xi}\right)\frac{\rho u^*}{\rho_1}, \qquad (6{\cdot}7a)$$

and in terms of the variables (6·2), (6·6) equations (3·13 b, c) become

$$\frac{\partial}{\partial \eta^*}\left(\frac{\mu}{\mu_1}\frac{\partial u^*}{\partial \eta^*}\right) = 4\xi G(\xi) + 4\frac{\rho u^*}{\rho_1}\xi\frac{\partial u^*}{\partial \xi} - 2w\frac{\partial u^*}{\partial \eta^*}, \qquad (6{\cdot}7b)$$

$$\frac{1}{\sigma_1}\frac{\partial}{\partial \eta^*}\left\{\frac{\sigma_1}{\sigma}\frac{\mu}{\mu_1}\frac{\partial}{\partial \eta^*}\left(\frac{\rho_1}{\rho}\right)\right\} + (\gamma-1)\mathscr{M}_1^2\frac{\mu}{\mu_1}\frac{p_1}{p}\left(\frac{\partial u^*}{\partial \eta^*}\right)^2 = 4\mathscr{M}_1^2\frac{p_1}{p}u^*\xi G(\xi) + 4\frac{\rho u^*}{\rho_1}\xi\frac{\partial}{\partial \xi}\left(\frac{\rho_1}{\rho}\right) - 2w\frac{\partial}{\partial \eta^*}\left(\frac{\rho_1}{\rho}\right). \qquad (6{\cdot}7c)$$

The terms arising directly from the pressure-gradient terms in (3·13b) and (3·13c) are $4\xi G(\xi)$ in (6·7b) and $4\mathscr{M}_1^2\frac{p_1}{p}u^*\xi G(\xi)$ in (6·7c). For zero pressure gradient these terms are absent and p_1/p in the second term on the left-hand side of (6·7c) is unity. The equations then assume a form in which the $\partial/\partial\xi$ terms are also absent so that they reduce, for constant σ, to

$$\left.\begin{aligned}
&\frac{dw}{d\eta^*} = \frac{\rho}{\rho_1}u^*, && (a)\\
&\frac{d}{d\eta^*}\left(\frac{\mu}{\mu_1}\frac{du^*}{d\eta^*}\right) = -2w\frac{du^*}{d\eta^*}, && (b)\\
&\frac{1}{\sigma_1}\frac{d}{d\eta^*}\left\{\frac{\mu}{\mu_1}\frac{d}{d\eta^*}\left(\frac{\rho_1}{\rho}\right)\right\} + (\gamma-1)\mathscr{M}_1^2\frac{\mu}{\mu_1}\left(\frac{du^*}{d\eta^*}\right) = -2w\frac{d}{d\eta^*}\left(\frac{\rho_1}{\rho}\right). && (c)
\end{aligned}\right\} \qquad (6{\cdot}8)$$

These are equations (5), (6) and (7) of Emmons & Brainerd (1942) in different notation, the relation between the notations being as follows:

Here	μ/μ_1	ρ_1/ρ	$2\eta^*$	u^*	v^*	$2w$
Emmons & Brainerd (1942)	ϕ	$\theta = T/T_1$	η	U	V	ξ/θ

the second of these relationships following from the fact that p is constant in Emmons & Brainerd (1942) so that $T/T_1 = \rho_1/\rho$.

A further change of independent variable simplifies considerably the numerical or mechanical integration both of the set of equations (6·8) for zero pressure gradient and, at least for constant σ, that of the more general equations (6·7). This is the use of

$$\eta \equiv \int_0 \frac{\mu_1}{\mu}\,d\eta^* \qquad (6{\cdot}9)$$

as an independent variable in place of η^*, the integration in (6·9) being understood to be carried out at constant ξ. This simplifies the highest derivative term in (6·7 b) and also, for constant σ, that in (6·7c), with important results in the later treatment of these equations.

Then
$$\frac{\mu}{\mu_1}\left(\frac{\partial}{\partial\eta^*}\right)_\xi = \left(\frac{\partial}{\partial\eta}\right)_\xi \tag{6·10}$$

and
$$\left(\frac{\partial}{\partial\xi}\right)_\eta = \left(\frac{\partial}{\partial\xi}\right)_{\eta^*} + \left(\frac{\partial\eta^*}{\partial\xi}\right)_\eta \left(\frac{\partial}{\partial\eta^*}\right)_\xi,$$

so that from (6·10)
$$\frac{\mu}{\mu_1}\left(\frac{\partial}{\partial\xi}\right)_{\eta^*} = \frac{\mu}{\mu_1}\left(\frac{\partial}{\partial\xi}\right)_\eta - \left(\frac{\partial\eta^*}{\partial\xi}\right)_\eta \left(\frac{\partial}{\partial\eta}\right)_\xi, \tag{6·11}$$

where
$$\eta^* \equiv \int_0 \frac{\mu}{\mu_1}\,d\eta, \tag{6·12}$$

the integration being carried out at constant ξ. Also

$$\frac{\mu}{\mu_1}\left\{\frac{2\rho u^*}{\rho_1}\xi\left(\frac{\partial}{\partial\xi}\right)_{\eta^*} - w\frac{\partial}{\partial\eta}\right\} = \frac{2\mu}{\mu_1}\frac{\rho u^*}{\rho_1}\xi\left(\frac{\partial}{\partial\xi}\right)_\eta - \left(w + \frac{2\rho u^*}{\rho_1}\xi\frac{\partial\eta^*}{\partial\xi}\right)\frac{\partial}{\partial\eta}. \tag{6·13}$$

Further, for $\mathscr{M}_1 = 0$ it is clear that equation (6·7c) has the solution $\rho_1/\rho = 1$, and the form of the equations suggests that for $\mathscr{M}_1 \neq 0$ the value of $(\rho_1/\rho - 1)$ at any point is of order $\mathscr{M}_1^2$. Hence it seems convenient to define a function r by

$$\frac{\rho_1}{\rho} = 1 + \mathscr{M}_1^2 r, \tag{6·14}$$

though, for a reason which will appear later, this substitution will not for the present be made in the combination $\rho u^*/\rho_1$. It will appear later (§ 11) that for $\mathscr{M}_1^2 = 0$ the solution of the equation for r is finite, which justifies this step.

Formula (6·13) suggests the use of the quantity

$$W = w + 2\frac{\rho u^*}{\rho_1}\xi\frac{\partial\eta^*}{\partial\xi} \tag{6·15}$$

in the place of w. From equations (6·7a) and (6·11) it follows that it satisfies the equation

$$\frac{\partial W}{\partial\eta} = \left(1 + 2\xi\frac{\partial}{\partial\xi}\right)\left(\frac{\mu}{\mu_1}\frac{\rho u^*}{\rho_1}\right), \tag{6·16a}$$

and in terms of η as independent variable, equations (6·7 b, c) become

$$\left.\begin{aligned} \frac{\partial^2 u^*}{\partial\eta^2} &= 4\xi\frac{\mu}{\mu_1}G(\xi) + \frac{4\rho u^*}{\rho_1}\frac{\mu}{\mu_1}\xi\frac{\partial u^*}{\partial\xi} - 2W\frac{\partial u^*}{\partial\eta}, & (b)\\ \frac{1}{\sigma_1}\frac{\partial}{\partial\eta}\left(\frac{\sigma_1}{\sigma}\frac{\partial r}{\partial\eta}\right) + (\gamma-1)\frac{p_1}{p}\left(\frac{\partial u^*}{\partial\eta}\right)^2 &= 4\xi\frac{\mu}{\mu_1}\frac{p_1}{p}u^*G(\xi) + \frac{4\rho u^*}{\rho_1}\frac{\mu}{\mu_1}\xi\frac{\partial r}{\partial\xi} - 2W\frac{\partial r}{\partial\eta}, & (c) \end{aligned}\right\} \tag{6·16}$$

or, for constant σ,

$$\frac{1}{\sigma_1}\frac{\partial^2 r}{\partial\eta^2} + (\gamma-1)\frac{p_1}{p}\left(\frac{\partial u^*}{\partial\eta}\right)^2 = 4\xi\frac{\mu}{\mu_1}\frac{p_1}{p}u^*G(\xi) + \frac{4\rho u^*}{\rho_1}\frac{\mu}{\mu_1}\xi\frac{\partial r}{\partial\xi} - 2W\frac{\partial r}{\partial\eta}. \tag{6·16c'}$$

Equations (6·16) could, of course, have been obtained from equations (3·13) by the transformation

$$\xi = U_1' x/U_1, \quad \eta = \tfrac{1}{2}(U_1/\nu_1 x)^{\frac{1}{2}}\int_0 (\mu_1/\mu)\,dy,$$

but the algebra of this direct transformation is somewhat tricky, and the point of the use of η rather than η^* is more apparent from the equations in the form (6·7) than from the form (3·13).

If ψ is the stream function defined by (4·7), then equation (6·16a) gives

$$W = (U_1'\xi/\nu_1)^{\frac{1}{2}}\,\partial\psi/\partial\xi. \tag{6·17}$$

For zero pressure gradient, when the terms involving ξ and $\partial/\partial\xi$ are absent, these equations reduce to

$$\left.\begin{aligned} &\frac{dw}{d\eta} = \frac{\mu}{\mu_1}\frac{u^*}{1+\mathscr{M}_1^2 r}, && (a)\\ &\frac{d^2u^*}{d\eta^2} = -2w\frac{du^*}{d\eta}, && (b)\\ &\frac{1}{\sigma}\frac{d^2r}{d\eta^2}+(\gamma-1)\left(\frac{du^*}{d\eta}\right)^2 = -2w\frac{dr}{d\eta}. && (c)\end{aligned}\right\} \tag{6·18}$$

Since η does not occur explicitly in (6·18) their order can be depressed by a change of independent variable, most conveniently, probably, to u^* since they can be written

$$\left.\begin{aligned} &\frac{d}{du^*}\left(\frac{du^*}{d\eta}\right) = -2w, && (a)\\ &\frac{d^2r}{du^{*2}}+\sigma(\gamma-1) = 2(1-\sigma)\,w\frac{dr}{du^*}\frac{d\eta}{du^*}, && (b)\end{aligned}\right\} \tag{6·19}$$

It is possible that this depression of the order of the equations would simplify the problem of satisfying the two-point boundary conditions, but otherwise it does not seem that this transformation has much practical advantage, except in the special case $\sigma = 1$ when the term on the right-hand side of (6·19b) disappears and the equation can be integrated formally; this case has already been considered by Busemann and by von Karman & Tsien.

For the equations for a non-zero pressure gradient, in the form (6·7) or (6·16), there are several methods of approach, which will be considered further in § 8.

7. The relation between the laminar boundary-layer equations for a compressible and an incompressible fluid

Bernoulli's equation shows that for $\mathscr{M}_1^2 \ll 1$, and for a given variation of U/U_1 in the main stream (and therefore a given relation between x and ξ), $\partial p/\partial x$ is of order U_1^2, so that $G(\xi)$ is $O(1)$ in $\mathscr{M}_1^2$. Then in the limit $\mathscr{M}_1^2 \to 0$, the terms in $\mathscr{M}_1^2$ in (6·7c) disappear, and this equation has then the solution

$$\rho_1/\rho = \text{constant} = 1.$$

Then the factor ρ/ρ_1 disappears in equations (6·7 a, b) and these then become

$$\left.\begin{aligned} &\frac{\partial w}{\partial\eta^*} = \left(1+2\xi\frac{\partial}{\partial\xi}\right)u^*, && (a)\\ &\frac{\partial}{\partial\eta^*}\left(\frac{\mu}{\mu_1}\frac{\partial u^*}{\partial\eta^*}\right) = 4\xi G(\xi)+4u^*\xi\frac{\partial u^*}{\partial\xi}-2w\frac{\partial u^*}{\partial\eta^*}, && (b)\end{aligned}\right\} \tag{7·1}$$

which are the laminar boundary-layer equations for incompressible flow, without restriction to uniform viscosity and with $2w$ in place of the function written f in the treatment of the corresponding equations for constant viscosity.

For an incompressible fluid whose viscosity is a function of temperature, a third equation would be necessary in order to give the temperature variation in the boundary layer. This equation can *not*, of course, be obtained from (6·7*c*), since in deriving this equation the equation of state of a perfect gas has been used in order to substitute both for the temperature T in terms of p and ρ, and also for C_p in terms of γ and the gas constant R, and neither of these substitutions is valid for an incompressible fluid. It would be necessary to go back to the energy equation in the basic form (3·10). With this understanding, then, regarding the treatment of the variation of viscosity with temperature if appreciable, we can say that the continuity and momentum equations for the laminar boundary layer in an incompressible fluid are the limits as $\mathscr{M}_1^2 \to 0$ of the corresponding equations for a compressible fluid.

For the flow of a perfect gas, it follows from (6·14), (6·16*c*) that if for $\mathscr{M}_1^2 = 0$ equation (6·16*c*) has a solution, say $r^{(0)}$, satisfying the boundary conditions, as is the case (see § 11 below) then, for $\mathscr{M}_1^2 \ll 1$, (ρ/ρ_1) is given, to first order in $\mathscr{M}_1^2$, by

$$\rho_1/\rho = 1 + \mathscr{M}_1^2 r^{(0)}.$$

The energy equation (6·18*c*) for zero pressure gradient only involves $\mathscr{M}_1^2$ through the term on the right-hand side, and it could be hoped that it would be possible to express its solution in a form which only varied slowly with $\mathscr{M}_1^2$. This partially explains why Pohlhausen's solution $I + \frac{1}{2}\sqrt{\sigma}\,(u^2+v^2) = \text{const.}$ (see § 3), of the incompressible form of the equation has been found to be a useful approximation for $\mathscr{M}_1^2 \leqslant 10$.

A full discussion of the energy equation is outside the scope of this paper. The following points seem, however, to deserve mention here. First, for liquids the Prandtl number σ is a function of temperature, so that the omission of the factor σ/σ_1 may not be justified. Secondly, for different liquids σ may have values from about $\frac{1}{30}$ (mercury) to about 200 (heavy oil) instead of being of order unity as it is for gases, so the velocity and temperature layers may not be of the same orders of thickness, and it should not be assumed without close examination that the surviving terms of the full equation (3·9) will be those of equation (3·10). Thirdly, cases may arise in which it becomes necessary to take into account not only the variation of viscosity of a liquid with temperature but also its variation of density with temperature, although the variation of density with pressure may be negligible. In such cases the 'compressible' equations of continuity and momentum (6·16 *a*, *b*) would be applicable but the form (6·16*c*) of the energy equation would not, and the latter would have to be replaced by one appropriate to the equation of state of the liquid in the temperature and pressure range contemplated. Further, the appropriate system of approximate equations might be other than the equations of the laminar boundary layer; in lubrication problems, for example, a 'slow motion' approximation is usually adopted.

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PART II. ALGEBRAIC PREPARATION OF THE EQUATIONS FOR SOLUTION

8. The equations for non-zero pressure gradient

For the case of a non-zero pressure gradient, the following seem to be possible methods of attack:

(*a*) Following Howarth, to attempt to find a solution in the form of expansions in ξ, with coefficients which are functions of η. The equations for these coefficients will be ordinary differential equations and it will be found (see §§ 8, 9 below) that, as in the incompressible case, they can be solved in succession rather than simultaneously, since the equations for the coefficients of ξ^n only involve the coefficients of ξ^j $(j \leqslant n)$; it is this feature which makes the evaluation of solutions by this method practicable. The equations to be solved are in effect fifth order linear (except for $n = 0$ when they are non-linear) with three boundary conditions given at one end of the range and two at the other. Although the accuracy finally required may be greater than that attainable by the differential analyzer, an exploratory treatment by this machine might be very valuable to find approximate solutions satisfying the two-point boundary conditions. Once approximate solutions were found it would probably be comparatively easy to improve them by an iterative numerical process (see § 12).

(*b*) To apply the differential analyzer to equations (6·7) using finite difference in ξ and integrating in η, following the general method suggested by Hartree & Womersley (1937) and applied to the equations of heat flow (Copple, Hartree, Porter & Tyson 1938), which has also been applied to incompressible boundary layer flow.

(*c*) To use finite differences in both ξ and η. One example of such a method has recently been proposed and applied to boundary layer flow by Cowling (1947), and another has been explored, in connexion with heat conduction in a solid in which a chemical reaction is taking place at a rate depending on the temperature and with evolution of heat, by Crank & Nicolson (1946). Such a method applied to equations (6·7) would probably be best carried out on high-speed automatic computing equipment using electro-magnetic relays or electronic counting circuits.

Even if one of methods (*b*) or (*c*) were used for the main part of the calculation, it would probably be found best to use method (*a*) to start the work, so as to get away from the singularity at the leading edge of the plate before picking up the integration by methods (*b*) or (*c*). This suggests that results of method (*a*) are likely to be required in any case, and the remainder of this paper is concerned with the derivation and solution of the relevant equations.

Emmons & Brainerd (1942) shows that the inclusion of the variation of viscosity with temperature has a considerable effect on the calculated values of C_f; hence although this complicates the problem appreciably, it will be taken into account from the beginning. It will be assumed that σ is constant, so that the form (6·15*c*) of the energy equation applies and the subscript 1 will be dropped.

It would be possible to use expansions in powers of ξ with coefficients which are functions either of η^* or of η. In the former case the differential equations for the functions of η^* would be obtained by substitution in the equations in the form (6·7); in the latter case the differential equations for the functions of η would be obtained by substitution in the equations in the form (6·16). At first sight there does not seem much to choose between these two possibilities.

But trial soon shows that the form of the highest derivative terms in the equations of momentum and energy in the forms (6·7 b, c) respectively leads, when series expansions of this kind are used, to complications which, though they do not appear serious algebraically, do become serious in numerical work on the resulting equations; and it seems likely that corresponding difficulties will appear in the other methods suggested for handling these equations by numerical or mechanical means. On the other hand, the highest order terms in the equations in the form (6·16), are of the simplest possible form; the independent variable (6·9) has been chosen precisely for the purpose of making them so. This introduces a really substantial simplification into the treatment of these equations by numerical or mechanical process, and it seems advisable to accept the slight complication of the form of the equation of continuity and of the lower order terms in the other equations in order to achieve this simplification of the highest order terms.

Further, the numerical results for zero pressure gradient (see § 19 below) show that $2u^*$ and r vary very much less with $\mathscr{M}_1^2$ at constant η than at constant η^*, so that by expressing results in terms of η rather than η^* as variable normal to the solid surface, fewer separate solutions need to be evaluated to give a set covering a given range of $\mathscr{M}_1^2$. It was hoped that the same would be true for the case of constant (but non-zero) pressure gradient; and since in this case the evaluation of a single solution is a major computing operation, means of minimizing the number of solutions to be evaluated is much more important, but this hope was not fulfilled.

The further development will be restricted to the case of uniform pressure gradient in the main stream, that is to say

$$G(\xi) \equiv 1, \tag{8·1}$$

so that, by integration of (6·3)

$$\frac{p}{p_1} = 1 + \gamma \mathscr{M}_1^2 \xi. \tag{8·2}$$

Also it will be assumed that the viscosity depends on the temperature T only, and that this relation is of the form

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^\beta = \left\{\left(\frac{p}{p_1}\right)\left(\frac{\rho_1}{\rho}\right)\right\}^\beta \tag{8·3}$$

with β constant. For the pressure variation (8·2), this gives

$$\frac{\mu}{\mu_1} = \{(1+\gamma \mathscr{M}_1^2 \xi)\,(1+\mathscr{M}_1^2 r)\}^\beta \tag{8·4}$$

on using (6·14).

Since equations (6·16) are three simultaneous equations for three dependent variables it would seem at first sight only necessary to introduce three expansions, for example

$$2u^* = \Sigma_n(-8\xi)^n\, h_n(\eta), \tag{8·5}$$

$$2W = \Sigma_n(-8\xi)^n\,(2n+1) f_n(\eta), \tag{8·6}$$

$$r = \Sigma_n(-8\xi)^n\, r_n(\eta). \tag{8·7}$$

Then it follows from (6·16 a) and (8·6) that

$$2\frac{\mu}{\mu_1}\frac{\rho u^*}{\rho_1} = \Sigma_n(-8\xi)^n f_n'(\eta) \tag{8·8}$$

and stream function ψ is given by

$$\psi = (\nu_1 \xi / U_1')^{\frac{1}{2}} \Sigma_n (-8\xi)^n f_n(\eta);$$

but it seems more convenient to introduce at this stage two additional expansions, namely

$$\frac{2\rho u^*}{\rho_1} = \Sigma_n (-8\xi)^n \chi_n(\eta), \tag{8·9}$$

and

$$\frac{\mu}{\mu_1} = \Sigma_n (-8\xi)^n \phi_n(\eta), \tag{8·10}$$

and possibly to eliminate the functions forming the coefficients of these expansions at a later stage. The use of expansions in (8ξ), of expansions for $2u^*$ and $2W$, and of the coefficients $(2n+1)$ in the latter, follows the usage of Howarth; for Howarth's case $\rho_1/\rho = 1$, and there is no distinction between the variables η and η^*, nor between $\frac{\mu}{\mu_1}\frac{\rho u^*}{\rho_1}$, $\frac{\rho u^*}{\rho_1}$, and u^*, so that the functions f_n are then the same as Howarth's f_n (or rather, they would be the same if Howarth had taken a constant pressure gradient in the main stream instead of a constant velocity gradient), and the functions h_n are Howarth's f_n' (with the same proviso).

There are five sets of relations between the five sets of functions f_n, r_n, h_n, ϕ_n, χ_n. Two sets of these relations are purely algebraical, the remaining three are in the form of differential equations.

From the relation
$$2u^* = \frac{\rho_1}{\rho}\frac{2\rho u^*}{\rho_1} = (1+\mathscr{M}_1^2 r)\frac{2\rho u^*}{\rho_1}$$

it follows that one of the two sets of algebraic relations is

$$\left.\begin{aligned} h_0 &= (1+\mathscr{M}_1^2 r_0)\,\chi_0, && \text{(i)}\\ h_1 &= (1+\mathscr{M}_1^2 r_0)\,\chi_1 + \mathscr{M}_1^2 r_1 \chi_0, && \text{(ii)}\\ h_2 &= (1+\mathscr{M}_1^2 r_0)\,\chi_2 + \mathscr{M}_1^2 (r_1\chi_1 + r_2\chi_0), && \text{(iii)}\end{aligned}\right\} \tag{8·11}$$

and in general

$$h_n = (1+\mathscr{M}_1^2 r_0)\,\chi_n + \mathscr{M}_1^2 \sum_{j=1}^{n} r_j \chi_{n-j}, \tag{8·12}$$

so that

$$\chi_n = \Big(h_n - \mathscr{M}_1^2 \sum_{j=1}^{n} r_j \chi_{n-j}\Big)\Big/(1+\mathscr{M}_1^2 r_0). \tag{8·13}$$

From (8·8), (8·9) and (8·10) another set of relations is

$$\left.\begin{aligned} f_0' &= \phi_0\chi_0, && \text{(i)}\\ f_1' &= \phi_1\chi_0 + \phi_0\chi_1, && \text{(ii)}\\ f_2' &= \phi_2\chi_0 + \phi_1\chi_1 + \phi_0\chi_2, && \text{(iii)}\end{aligned}\right\} \tag{8·14}$$

and in general

$$f_n' = \sum_{j=0}^{n} \phi_j \chi_{n-j}. \tag{8·15}$$

Elimination of the functions χ between (8·13) and (8·15), or more simply use of (8·5), (8·7) and (8·10), gives

$$f_n' = \Big\{\sum_{j=0}^{n} \phi_j h_{n-j} - \mathscr{M}_1^2 \sum_{j=0}^{n-1} f_j' r_{n-j}\Big\}\Big/\{1+\mathscr{M}_1^2 r_0\}. \tag{8·16}$$

Only two of the sets of relations (8·13), (8·15) and (8·16) are independent, but all are useful at different stages of the work.

For the pressure variation (8·2), it follows from (8·4) and (8·7) that

$$\mu/\mu_1 = (1+\mathscr{M}_1^2 r_0)^\beta \{1-(8\xi)\,s_1 + (8\xi)^2 s_2 - (8\xi)^3 s_3 + \ldots\}^\beta \tag{8·17}$$

where
$$\left.\begin{aligned} s_1 &= \frac{\mathscr{M}_1^2 r_1}{1+\mathscr{M}_1^2 r_0} - m, & \text{(i)}\\ s_j &= \mathscr{M}_1^2 (r_j - m r_{j-1})/(1+\mathscr{M}_1^2 r_0) \quad (j \geqslant 2) & \text{(ii)} \end{aligned}\right\} \tag{8·18}$$
$$m = \tfrac{1}{8}\gamma \mathscr{M}_1^2. \tag{8·19}$$

If (8·17) is expanded and compared with (8·10), the following relations between the coefficients in the expansions (8·7) and (8·10) are obtained:
$$\left.\begin{aligned} \phi_0 &= (1+\mathscr{M}_1^2 r_0)^\beta, & \text{(i)}\\ \phi_1 &= \beta\phi_0 s_1, & \text{(ii)}\\ \phi_2 &= \beta\phi_0\{s_2 - \tfrac{1}{2}(1-\beta)\, s_1^2\}, & \text{(iii)}\\ \phi_3 &= \beta\phi_0\{s_3 - (1-\beta)\, s_1 s_2 - \tfrac{1}{6}(1-\beta)\,(2-\beta)\, s_1^3\}, & \text{(iv)} \end{aligned}\right\} \tag{8·20}$$
and in general $\phi_n = \beta_0 \times [\text{coefficient of } X^n \text{ in expansion of } (\Sigma_j s_j X^j)^\beta]$. (8·21)

It is convenient to write this
$$\phi_n = \beta\phi_0 (s_n - S_n) \tag{8·22}$$
so that S_n involves only functions r_j with $j < n$.

Then from substitution in (8·16) it follows that
$$f_n' = \left\{h_n - \frac{(1-\beta)\mathscr{M}_1^2 h_0}{1+\mathscr{M}_1^2 r_0} r_n\right\} \Big/ (1+\mathscr{M}_1^2 r_0)^{1-\beta} - C_n, \tag{8·23}$$
where C_n involves only functions of order $j < n$; in particular
$$\left.\begin{aligned} f_0' &= h_0/(1+\mathscr{M}_1^2 r_0)^{1-\beta}, & \text{(i)}\\ f_1' &= \left\{h_1 - \frac{(1-\beta)\mathscr{M}_1^2 h_0}{1+\mathscr{M}_1^2 r_0} r_1 - \beta m h_0\right\}\Big/ (1+\mathscr{M}_1^2 r_0)^{1-\beta}. & \text{(ii)} \end{aligned}\right\} \tag{8·24}$$

Equations (8·23) form one set of differential equations between the five sets of functions h_n, r_n, f_n, ϕ_n, χ_n. The other two sets are obtained by substitution of the series (8·5) to (8·8) and (8·10) into equations (6·16 b, c). From equation (6·16 b),
$$\left.\begin{aligned} h_0'' &= -f_0 h_0', & \text{(i)}\\ h_1'' &= -\phi_0 - f_0 h_1' - 3f_1 h_0' + 2f_0' h_1, & \text{(ii)}\\ h_2'' &= -\phi_1 - f_0 h_2' - 3f_1 h_1' - 5f_2 h_0' + 2(f_1' h_1 + 2f_0' h_2), & \text{(iii)} \end{aligned}\right\} \tag{8·25}$$
and in general
$$h_n'' = -\phi_{n-1} - \sum_{j=0}^{n} (2j+1) f_j h_{n-j}' + \sum_{j=0}^{n} 2j h_j f_{n-j}'. \tag{8·26}$$
And from equation (6·16 c)
$$\left.\begin{aligned} &(1/\sigma)\, r_0'' + \tfrac{1}{4}(\gamma-1)\,(h_0')^2 = -f_0 r_0', & \text{(i)}\\ &(1/\sigma)\, r_1'' + \tfrac{1}{4}(\gamma-1)\,\{2h_0' h_1' + m(h_0')^2\} = -\tfrac{1}{4}\phi_0 h_0 - f_0 r_1' - 3f_1 r_0' + 2f_0 r_1', & \text{(ii)}\\ &(1/\sigma)\, r_2'' + \tfrac{1}{4}(\gamma-1)\,\{2h_0' h_2' + (h_1')^2 + m(2h_0' h_1') + m^2 (h_0')^2\}\\ &\qquad = -\tfrac{1}{4}\{(\phi_0 h_1 + \phi_1 h_0) + m\phi_0 h_0\} + 3f_1 r_1' - 5f_2 r_0' + 2(f_1' r_1 + 2f_0' r_2), & \text{(iii)} \end{aligned}\right\} \tag{8·27}$$
and in general
$$(1/\sigma)\, r_n'' + \tfrac{1}{4}(\gamma-1)\left\{\sum_{k=0}^{n} m^k \sum_{j=0}^{n-k} h_j' h_{n-k-j}'\right\}$$
$$= -\frac{1}{4}\sum_{k=0}^{n-1} m^k \sum_{j=0}^{n-k-1} \phi_j h_{n-k-j-1} - \sum_{j=0}^{n} (2j+1) f_j r_{n-j}' + \sum_{j=0}^{n} 2j r_j f_{n-j}'. \tag{8·28}$$

9. General structure of the set of equations

These equations look elaborate at first sight, but on further examination it seems that they may not be quite so formidable as they first appear.

It is convenient to refer to the set of five functions f_n, h_n, r_n, ϕ_n, χ_n, with any given value of n, as the 'nth order functions'. One feature of the set of equations is that the equations for the nth order functions, that is to say the equations involving the highest derivatives of these functions in the differential equations, involve only the jth order functions for $j \leqslant n$, so that the equations for these sets of five functions can be solved *successively* in the order of n increasing; the equations for different values of n do not all have to be solved simultaneously. Then, in evaluating the solutions of the equations for the nth order functions, the functions of lower order will be known functions of η; this will be taken as understood in the following.

Further, with this understanding, the equations for the nth order functions ($n>0$) are linear and all of the same general form, namely

$$f'_n = \left\{h_n - \frac{(1-\beta)\mathcal{M}_1^2 h_0}{1+\mathcal{M}_1^2 r_0} r_n\right\} \Big/ (1+\mathcal{M}_1^2 r_0)^{1-\beta} - C_n. \tag{8·23}$$

$$h''_n = -f_0 h'_n + 2n f'_0 h_n - (2n+1) f_n h'_0 - D_n, \tag{9·1}$$

$$(1/\sigma)\, r''_n + \tfrac{1}{2}(\gamma-1)\, h'_0 h'_n = -f_0 r'_n + 2n f'_0 r_n - (2n+1) f_n r'_0 - E_n, \tag{9·2}$$

where C_n, D_n, E_n involve only the lower-order functions which will be known as functions of η in the evaluation of the solution of these equations in the order of n increasing. The expressions for C_n, D_n and E_n are

$$C_n = \left[\beta\phi_0 h_0 \left\{S_n + \frac{m\mathcal{M}_1^2 r_{n-1}}{1+\mathcal{M}_1^2 r_0}\right\} - \sum_{j=1}^{n-1} (h_j \phi_{n-j} - \mathcal{M}_1^2 f'_j r_{n-j})\right] \Big/ (1+\mathcal{M}_1^2 r_0), \tag{9·3}$$

$$D_n = \phi_{n-1} + \sum_{j=1}^{n-1} \{(2j+1) f_j h'_{n-j} - 2j h_j f'_{n-j}\}, \tag{9·4}$$

$$E_n = \frac{1}{4}\sum_{k=0}^{n-1} m^k \sum_{j=0}^{n-k-1} \phi_j h_{n-k-j-1} + \tfrac{1}{4}(\gamma-1)\left\{\sum_{j=1}^{n-1} h'_j h'_{n-j} + \sum_{k=1}^{n} m^k \sum_{j=0}^{n-k} h'_j h'_{n-k-j}\right\} + \sum_{j=1}^{n-1}\{(2j+1) f_j r'_{n-j} - 2j r_j f'_{n-j}\}. \tag{9·5}$$

(Note: formula (9·3) for C_n applies for $n \geqslant 2$; for $n=1$ see formula (8·24 ii) above).

The main complication in the calculation of the higher order functions is not in the solution of the equations but in the evaluation of the functions C_n, D_n and E_n.

The equations for the null-order functions ($n=0$) are exceptional in being non-linear. They are the first equations of each of the sets (8·20), (8·24), (8·25) and (8·27), namely

$$\left.\begin{aligned} \phi_0 &= (1+\mathcal{M}_1^2 r_0)^{\beta}, && (8·20\,\text{i}) \\ f'_0 &= h_0/(1+\mathcal{M}_1^2 r_0)^{1-\beta}, && (8·24\,\text{i}) \\ h''_0 &= -f_0 h'_0, && (8·25\,\text{i}) \\ (1/\sigma)\, r''_0 + \tfrac{1}{4}(\gamma-1)\,(h'_0)^2 &= -f_0 r'_0. && (8·27\,\text{i}) \end{aligned}\right\}$$

Equations (8·24i), (8·25i) and (8·27i) are equations (6·16) for the case of zero pressure gradient, with the substitutions

$$2u^* = h_0, \quad 2w = f_0, \quad r = r_0$$

and with the viscosity variation (8·3). The numerical treatment of these equations is considered in § 12.

It will be seen in §10 that the boundary conditions for the solution of the equations for the null-order functions are

$$\left.\begin{aligned} h_0 = f_0 = r_0' = 0 \quad &\text{at } \eta = 0 \\ \text{and} \qquad h_0 = 2, \quad r_0 = 0 \quad &\text{at } \eta = \infty. \end{aligned}\right\} \tag{9·6}$$

If for shortness we write

$$q = \exp\left(-\int_0 f_0\, d\eta\right), \tag{9·7}$$

the solutions of (8·25i) and (8·27i) can be written in terms of q in the following way:

$$h_0' = h_0'(0)\, q = 2q\bigg/\int_0^\infty q\, d\eta, \tag{9·8}$$

$$h_0 = 2\int_0 q\, d\eta\bigg/\int_0^\infty q\, d\eta, \tag{9·9}$$

$$r_0' = -\tfrac{1}{4}(\gamma - 1)\, \sigma\{h_0'(0)\}^2\, q^\sigma \int_0 q^{2-\sigma}\, d\eta, \tag{9·10}$$

$$r_0 = \tfrac{1}{4}(\gamma - 1)\, \sigma\{h_0'(0)\}^2 \int^\infty q^\sigma \left\{\int_0 q^{2-\sigma}\, d\eta\right\} d\eta. \tag{9·11}$$

For large $\eta, f_0' \to 2$ and $f' - 2 = o(1/\eta)$ and hence

$$f_0 \sim 2(\eta - c). \tag{9·12}$$

Consequently

$$q = O(e^{-\eta^2}),$$

and from (9·8), (9·9)

$$\left.\begin{aligned} h_0 &= O(e^{-\eta^2}), \\ r_0' &= O(e^{-\sigma\eta^2}). \end{aligned}\right\} \tag{9·13}$$

The treatment of the equations for the higher order functions is considered in § 13.†

† [*Note added in proof*]. Dr L. Howarth has pointed out the considerable simplification which occurs in the equations if β is taken to have the value unity. Then, first, the null-order functions (with η as independent variable) are independent of the Mach number $\mathscr{M}_1$; secondly, the expansion (8·21) is much simplified, and gives just $\phi_n = \phi_0 s_n$, so that $S_n = 0$ for all n; and thirdly, the coefficient of r_n in equation (8·23) is zero, so that, as for the null-order functions with $\beta = 1$, r_n does not occur in the equations for f_n, h_n. This last is the most important simplification from the point of view of the practical procedure for the solution of the equations; it follows from it that the equations for f_n, h_n can be solved first, and the equations for r_n solved later with f_n, h_n then known functions of η; and the values of $h_n'(0)$ and $r_n(0)$ required to satisfy the conditions at infinity can be determined separately instead of having to be found simultaneously.

Since the value of β for air is not greatly different from unity, Dr Howarth suggests that the general character of the effect of compressibility on the laminar boundary layer, including the variation of viscosity with temperature (which is important when there is a retarding pressure gradient, see § 19) would probably be exhibited by solutions calculated for the simpler equations which apply when $\beta = 1$.

10. Boundary conditions

Since $u = v = 0$ everywhere on the solid boundary $y = 0$, it follows that $u^* = w = 0$ for $\eta = 0$, all ξ, and hence

$$h_n(0) = f_n(0) = 0, \quad \text{all } n. \tag{10·1}$$

Also for solutions referring to conditions in which there is no heat flow across the solid boundary, $\partial T/\partial y = 0$ everywhere on the solid boundary, and since $\partial p/\partial y = 0$ through the boundary layer, this implies $\partial(\rho_1/\rho)/\partial y = 0$ for $\eta = 0$, all ξ, whence

$$r_n'(0) = 0, \quad \text{all } n. \tag{10·2}$$

Equations (8·23), (9·1) and (9·2) for f_n, h_n, r_n are in effect fifth order, and two further boundary conditions are necessary to determine the solutions. These express the requirements that the velocity and density in the boundary layer should tend asymptotically to the values in the main stream; since $\partial p/\partial y = 0$ through the boundary layer, this ensures that the temperature also tends asymptotically to the main stream value. These conditions can be expressed in two forms. One form is that at each ξ the velocity and density should each tend to some finite asymptotic value as $\eta \to \infty$, that is not only that $h_n'(\infty) = r_n'(\infty) = 0$ but that

$$\eta h_n'(\eta) \to 0, \quad \eta r_n'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \quad \text{all } n; \tag{10·3}$$

the other form is the specification of the limiting values of h_n and r_n, which are given by fitting the variations of the values of velocity and density with ξ in the boundary layer to those in the main stream.

The flow in the main stream is adiabatic potential flow. Since the pressure distribution is taken to be $p/p_1 = 1 + \gamma \mathscr{M}_1^2 \xi$ (see (8·2)), it follows that the density distribution in the main stream is given by

$$1 + \mathscr{M}_1^2 r = (\rho_1/\rho) = (1 + \gamma \mathscr{M}_1^2 \xi)^{-1/\gamma}.$$

On expanding in powers of (8ξ) and comparing with (8·7) it follows that

$$\left.\begin{aligned} r_0(\infty) &= 0, && \text{(i)}\\ r_1(\infty) &= \tfrac{1}{8}, && \text{(ii)}\\ r_2(\infty) &= \frac{\gamma+1}{8\,.\,2!}\,(\tfrac{1}{8}\mathscr{M}_1^2), && \text{(iii)}\\ r_3(\infty) &= \frac{(\gamma+1)\,(2\gamma+1)}{8\,.\,3!}\,(\tfrac{1}{8}\mathscr{M}_1^2)^2, && \text{(iv)} \end{aligned}\right\} \tag{10·4}$$

or generally

$$r_n(\infty) = \frac{1}{\mathscr{M}_1^2}\,[\text{coefficient of } (8\xi)^n \text{ in expansion } \{1 + (\tfrac{1}{8}\gamma \mathscr{M}_1^2)\,(8\xi)\}^{-1/\gamma}]. \tag{10·5}$$

Also if $U = U_1 U^*(\xi)$ is the velocity in the main stream at ξ, Bernoulli's equation gives

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \tfrac{1}{2}U^2 = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \tfrac{1}{2}U_1^2,$$

whence

$$(U^*)^2 = 1 - \frac{2}{(\gamma-1)\,\mathscr{M}_1^2}\left\{\left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right\}. \tag{10·6}$$

The most convenient way of finding the expansion of U^* in the powers of 8ξ is to differentiate (10·6) repeatedly with respect to ξ and then to put $\xi=0$ in the successive derivatives. Since $p/p_1 = 1+\gamma\mathscr{M}_1^2\xi$ for the pressure distribution here considered, $\dfrac{\partial}{\partial\xi}\left(\dfrac{p}{p_1}\right)$ can be replaced by its constant value $\gamma\mathscr{M}_1^2$ in each differentiation. Hence we get in succession

$$\left.\begin{aligned} U^*\frac{dU^*}{d\xi} &= -\left(\frac{p}{p_1}\right)^{-1/\gamma}, \\ U^*\frac{d^2U^*}{d\xi^2}+\left(\frac{dU^*}{d\xi}\right)^2 &= \mathscr{M}_1^2\left(\frac{p}{p_1}\right)^{-[(1/\gamma)+1]}, \\ U^*\frac{d^3U^*}{d\xi^3}+3\frac{dU^*}{d\xi}\frac{d^2U^*}{d\xi^2} &= -(\gamma+1)\,\mathscr{M}_1^4\left(\frac{p}{p_1}\right)^{-[(1/\gamma)+2]}, \quad \text{etc.} \end{aligned}\right\} \tag{10·7}$$

Now at $\xi=0$, $U^*=1$ and $p/p_1=1$, hence

$$\left(\frac{dU^*}{d\xi}\right)_{\xi=0} = 1,\quad \left(\frac{d^2U^*}{d\xi^2}\right)_{\xi=0} = \mathscr{M}_1^2-1,\quad \left(\frac{d^3U^*}{d\xi^3}\right)_{\xi=0} = -\{(\gamma+1)\,\mathscr{M}_1^4-3\mathscr{M}_1^2+3\},\quad \text{etc.,} \tag{10·8}$$

and from (8·5)

$$h_0(\infty) = 2(U^*)_{\xi=0},\qquad h_1(\infty) = -\frac{2}{8}\left(\frac{dU^*}{d\xi}\right)_{\xi=0},$$

$$2!\,h_2(\infty) = \frac{2}{8^2}\left(\frac{d^2U^*}{d\xi^2}\right)_{\xi=0},\quad 3!\,h_3(\infty) = -\frac{2}{8^3}\left(\frac{d^3U^*}{d\xi^3}\right)_{\xi=0},\quad \text{etc.,}$$

and hence

$$\left.\begin{aligned} h_0(\infty) &= 2, && \text{(i)} \\ h_1(\infty) &= \tfrac{1}{4}, && \text{(ii)} \\ h_2(\infty) &= \tfrac{1}{64}(\mathscr{M}_1^2-1), && \text{(iii)} \\ h_3(\infty) &= \frac{1}{6.256}\{(\gamma+1)\,\mathscr{M}_1^4-3\mathscr{M}_1^2+3\}, \quad \text{etc.} && \text{(iv)} \end{aligned}\right\} \tag{10·9}$$

The conditions (10·3) are clearly much simpler than the conditions (10·4) and (10·9), particularly for the higher values of n. It will be shown that for $n\neq 0$ the two forms of the conditions are equivalent; for but $n=0$ this is not the case. For $n=0$, $f_0>0$ for any solution for which $f_0'(0)>0$, and consequently, from (9·7), (9·8) and (9·10) it follows that the conditions (10·3) are satisfied by any such solution, irrespective of the asymptotic values of h_0 and r_0. Hence for $n=0$ it is necessary to take the boundary conditions at infinity in the form

$$h_0(\infty) = 2,\quad r_0(\infty) = 0, \tag{10·10}$$

as quoted in §8.

Consider, on the other hand, the set of equations for the first-order functions,

$$f_1' = \left\{h_1 - \frac{(1-\beta)\,\mathscr{M}_1^2 h_0}{1+\mathscr{M}_1^2 r_0}r_1 - \beta m h_0\right\}\Big/(1+\mathscr{M}_1^2 r_0)^{1-\beta}, \tag{8·24 ii}$$

$$h_1'' = -\phi_0 - f_0 h_1' - 3f_1' h_0 + 2f_0' h_1, \tag{8·25 ii}$$

$$(1/\sigma)\,r_1'' + \tfrac{1}{4}(\gamma-1)\,\{2h_0'h_1' + m(h_0')^2\} = -\tfrac{1}{4}\phi_0 h_0 - f_0 r_1' - 3f_1 r_0' + 2f_0' r_1. \tag{8·27 iii}$$

Now $r_0(\infty)=0$, $h_0(\infty)=2$, $f_0\sim 2(\eta-c)$; and further h_0' is $o(e^{-\eta^2})$ and r_0' is $o(e^{-\sigma\eta^2})$, whereas f_1 is $O(\eta)$.

Hence the asymptotic forms of equations (8·25 ii) and (8·27 ii) are

$$h_1'' \sim -1-2(\eta-c)\,h_1'+4h_1, \tag{10·11}$$

$$(1/\sigma)\,r_1'' \sim -\tfrac{1}{2}-2(\eta-c)\,r_1'+4r_1. \tag{10·12}$$

Now if $\eta h_1' \to 0$ in accordance with condition (10·3), it follows from (10·11) that $h_1 \to \frac{1}{4}$, and similarly from (10·12) that $r_1 \to \frac{1}{8}$.

Hence the condition that h_1 and r_1 should each tend to *some* asymptotic value determines these values, namely

$$h_1(\infty) = \tfrac{1}{4}, \quad r_1(\infty) = \tfrac{1}{8},$$

which are just the values (10·8 ii) and (10·4 ii).

It will be seen that this result follows from the presence, in equations (8·25 ii) and (8·27 ii), of the terms $2f_0'h_1$ and $2f_0'r_1$, which are special cases of the terms $2nf_0'h_n$ and $2nf_0'r_n$ in the equations for the nth order functions. These terms will be present in the equations for all the higher order functions, so that the same situation will occur in them. On the other hand, the coefficients of these terms include a factor n, and they are therefore absent for $n = 0$, and the condition that h_0, r_0 should tend to asymptotic values no longer determines these values.

Further if Z is written for $(\eta-c)$, it follows from (9·13) that for large η

$$\frac{d^2}{dZ^2}(h_1-\tfrac{1}{4})+2Z\frac{d}{dZ}(h_1-\tfrac{1}{4})-4(h_1-\tfrac{1}{4}) = 0. \tag{10·13}$$

Now two solutions of the equation

$$\frac{d^2\zeta}{dZ^2}+2Z\frac{d\zeta}{dZ}-2n\zeta = 0$$

are (Hartree 1935) exactly, not only asymptotically, a polynomial of degree n in z (in fact the Hermite polynomial of order n and argument iz) and the n-fold integral of the error function, with integration constants chosen so that this integral is zero at infinity, which is of order e^{-Z^2}/Z^{n+1} for large Z. Since $h_1-\frac{1}{4}$ is zero at infinity, it follows that if it is satisfied (10·13) exactly, it would have this behaviour at infinity, but since terms of order $\eta e^{-\eta^2}$ have been omitted in equation (10·13) it follows that h_1 is of the same order.

This argument can be extended. It can be shown similarly that h_n' and r_n' are at most of order $\eta^p e^{-\sigma\eta^2}$, where p is finite, whereas f_n is at most of order η; hence the only terms in equations (8·25) remaining finite are those with ϕ_j as a factor, and these, on re-arranging, become (starting with (8·25 ii))

$$\phi_0(2\chi_0h_1-1) = 0,$$

$$2\phi_0(2\chi_0h_2+\chi_1h_1)+\phi_1(2\chi_0h_1-1) = 0,$$

$$2\phi_0(3\chi_0h_3+2\chi_1h_2+\chi_2h_1)+2\phi_1(\chi_1h_1+\chi_0h_2)+\phi_2(2\chi_0h_1-1) = 0,$$

etc., so that in succession we have the asymptotic relations

$$\left.\begin{aligned} 2\chi_0h_1-1 &= 0,\\ 2\chi_0h_2+\chi_1h_1 &= 0,\\ 3\chi_0h_3+2\chi_1h_2+\chi_2h_1 &= 0,\end{aligned}\right\} \tag{10·14}$$

etc. In like manner equations (8·27) become

$$\phi_0(2\chi_0 r_1 - \tfrac{1}{4}h_0) = 0,$$
$$\phi_0\{2(2\chi_2 r_0 + \chi_1 r_1) + \tfrac{1}{4}(mh_0 - h_1)\} + \phi_1(2\chi_0 r_1 - \tfrac{1}{4}h_0) = 0,$$

etc., so that in succession

$$\left.\begin{aligned} 2\chi_0 r_1 + \tfrac{1}{4}h_0 &= 0, \\ 2(2\chi_0 r_2 + \chi_1 r_1) + \tfrac{1}{4}(mh_0 - h_1) &= 0, \end{aligned}\right\} \tag{10·15}$$

etc. Since the asymptotic values of ϕ_j do not enter into the relations (10·14), (10·15), it is not necessary to use the expansions (8·20); this is to be expected, since the asymptotic values of h_n, r_n depend on the flow in the main stream, which is unaffected by viscosity, whereas the relations (8·20) depend on the relation between viscosity and temperature.

A third set of relations, however, is necessary before the asymptotic values of h_n, r_n can be found from (10·15); these are given by (8·13),

$$\chi_n = h_n - \sum_{j=0}^{n} r_j \chi_{n-j}, \tag{10·16}$$

since $r_0(\infty) = 0$.

The solution of (10·10), (10·14) and (10·15) in succession gives an alternative way of evaluating $h_n(\infty)$, $r_n(\infty)$ and $\chi_n(\infty)$.

11. The case $\mathscr{M}_1 = 0$

The case $\mathscr{M}_1 = 0$ is worth particular attention, both because of the comparative simplicity of the equations in this case, and the partial check on the general equations which they give by comparison with those for incompressible flow, and because the results in this case form a useful starting point for the solution of the equations for $\mathscr{M}_1^2 \neq 0$ by an iterative process.

For $\mathscr{M}_1 = 0$, the equations for the functions r_n (equations (10·4) below) certainly have finite solutions satisfying the boundary conditions, this justifies the assumption that $\{(\rho_1/\rho) - 1\}/\mathscr{M}_1^2$ remain finite for $\mathscr{M}_1 = 0$, on which the substitution (6·12) was based, and shows that, as was mentioned in § 7, the boundary layer flow for a compressible fluid tends to that for an incompressible fluid as $\mathscr{M}_1 \to 0$, so that a comparison of the equations for $\mathscr{M}_1 = 0$ with the equations for incompressible flow provide a check on the former.

For $\mathscr{M}_1 = 0$, equations (8·20) show that:

$$\phi_0 = 1, \quad \phi_n = 0 \quad (\text{all } n > 0), \tag{11·1}$$

and hence $\eta^* = \eta$, so that η-derivatives (indicated here by dashes) are equivalent to η^*-derivatives (indicated by dashes in incompressible flow). Also there is no distinction between $2u^*$ and $2\rho u^*/\rho_1$ so that from (8·13) $\chi_n = h_n$ (all n), and from (8·15) $f_n' = \chi_n$, so it follows that

$$\chi_n = h_n = f_n' \quad (\text{all } n). \tag{11·2}$$

Equations (7·19) become

$$\left.\begin{aligned} f_0''' &= -f_0 f_0'', && \text{(i)} \\ f_1''' &= -1 - (f_0 f_1'' + 3f_1 f_0'') + 2f_0' f_1', && \text{(ii)} \\ f_2''' &= -(f_0 f_2'' + 3f_1 f_1'' + 5f_2 f_0'') + 4f_0' f_2' + 2(f_1')^2, && \text{(iii)} \\ f_3''' &= -(f_0 f_3'' + 3f_1 f_2'' + 5f_2 f_1'' + 7f_3 f_0'') + 6f_0' f_3' + 4f_1' f_2', && \text{(iv)} \end{aligned}\right\} \tag{11·3}$$

with boundary conditions $f_n(0) = f_n'(0) = 0, \quad \text{all } n,$

$$f_0'(\infty) = 2, \quad f_1'(\infty) = \tfrac{1}{4}, \quad f_2'(\infty) = -\tfrac{1}{64}, \quad f_3'(\infty) = \tfrac{1}{512}, \quad \ldots,$$

(11·3 i, ii and iv) and subsequent equations are the same as Howarth's; (11·3 iii) and the boundary conditions at infinity for $n \geqslant 2$ are different because Howarth's equations and their solutions refer to the case of uniform *velocity* gradient in the main stream and not to uniform *pressure* gradient.

Equations (7·20) become

$$\left.\begin{aligned}(1/\sigma)\, r_0'' + \tfrac{1}{4}(\gamma-1)\,(f_0'')^2 &= -f_0 r_0', & \text{(i)}\\ (1/\sigma)\, r_1'' + \tfrac{1}{4}(\gamma-1)\,(2f_0''f_1'') &= -\tfrac{1}{4}f_0' - (f_0 r_1' + 3f_1 r_0') + 2f_0' r_1, & \text{(ii)}\\ (1/\sigma)\, r_1'' + \tfrac{1}{4}(\gamma-1)\,\{(f_1'')^2 + 2f_0''f_2''\} &= -\tfrac{1}{4}f_1' - (f_0 r_2' + 3f_1 r_1' + 5f_2 r_0') + 4f_0' r_2 + 2f_1' r_1, & \text{(iii)}\end{aligned}\right\} \quad (11·4)$$

with boundary conditions $r_n'(0) = 0, \quad \text{all } n$

$$r_0(\infty) = 0, \quad r_1(\infty) = \tfrac{1}{8}, \quad r_n(\infty) = 0 \quad (n \geqslant 2).$$

Equations (11·3) do not involve the functions r_n, which can be evaluated by the solution of equations (11·4) in succession *after* the solutions of equations (11·3) have been calculated, instead of simultaneously with them.

The solutions of (11·3 i, ii) have been given by Howarth; the solution of (11·4 i), with $\theta \equiv \dfrac{8}{\gamma-1} r_0$ as dependent variable, has been given by Pohlhausen (F.D. p. 628).

The general form of the equation for r_n is

$$(1/\sigma)\, r_n'' + f_0 r_n - 2nf_0 r_n = \overline{E}_n, \tag{11·5}$$

where $\overline{E}_n$ involves f_n as well as lower-order functions, but, for $\mathscr{M}_1 = 0$, can be regarded as known since in this case the equation for f_n does not involve r_n. Now f_0, f_0' have no singularities for finite positive η; hence the solutions of the homogeneous equation obtained by putting $\overline{E} = 0$ in (11·5) have no singularities in this range. A formal solution of (11·5) in terms of two solutions of the homogeneous equation can be written down by the method of 'variation of parameters', and although of little use for the practical evaluation of solutions, it shows that there is a solution of (11·5) satisfying the required boundary conditions and finite for all positive η, as quoted earlier in this section.

12. Evaluations of the null-order functions

The null-order functions can be evaluated very conveniently by iterative quadrature using the equations

$$f_0' = h_0/(1+\mathscr{M}_1^2 r_0)^{1-\beta}, \tag{8·24 i}$$

$$q = \exp\left(-\int_0 f_0\, d\eta\right), \tag{9·7}$$

$$h_0 = 2\int_0 q\, d\eta \Big/ \int_0^\infty q\, d\eta, \tag{9·9}$$

$$r_0 = \tfrac{1}{4}(\gamma-1)\,\{h_0'(0)\}^2 \int^\infty q^\sigma\left(\int_0 q^{2-\sigma} d\eta\right) d\eta. \tag{9·11}$$

Starting from estimates of the functions h_0 and r_0 (taking the solutions for $\mathscr{M}_1 = 0$ if no better estimates are available), f_0' is calculated from (8·24 i) and integrated twice to give $\int_0 f_0\, d\eta$; since f_0 itself is not required at this stage, this two-fold integration is conveniently done without calculating f, by double-summing the second difference of $\int_0 f_0\, d\eta$ given in central difference notation by

$$\delta^2\left(\int_0 f_0\, d\eta\right) = (\delta\eta)^2\,(f_0' + \tfrac{1}{12}\delta^2 f_0' - \tfrac{1}{240}\delta^4 f_0' \ldots).$$

From $\int_0 f_0\, d\eta$, q is obtained from (9·7) and h_0 from (9·9) after a further integration; then r_0 is evaluated from (9·11).

The iterative process can be illustrated by a block diagram, figure 6, where the 'blocks' represent the various quantities calculated. This shows that the iterative process can be regarded as having two 'loops', one, including the letter (A) in the figure, giving h_0 and the other, including the letter (B), giving r_0. The iteration process (A) for h_0, if r_0 is regarded as given, is much simpler than that for r_0, and for small values of $\mathscr{M}_1^2$ the result of a complete iteration, once round each loop, is much more sensitive to the estimate of h_0 than to that of r_0 (this is to be expected since for $\mathscr{M}_1^2 = 0$, h_0 is unaffected by r_0), and it may be then worth making two circuits of the iteration loop for h_0 for each estimate of r_0.

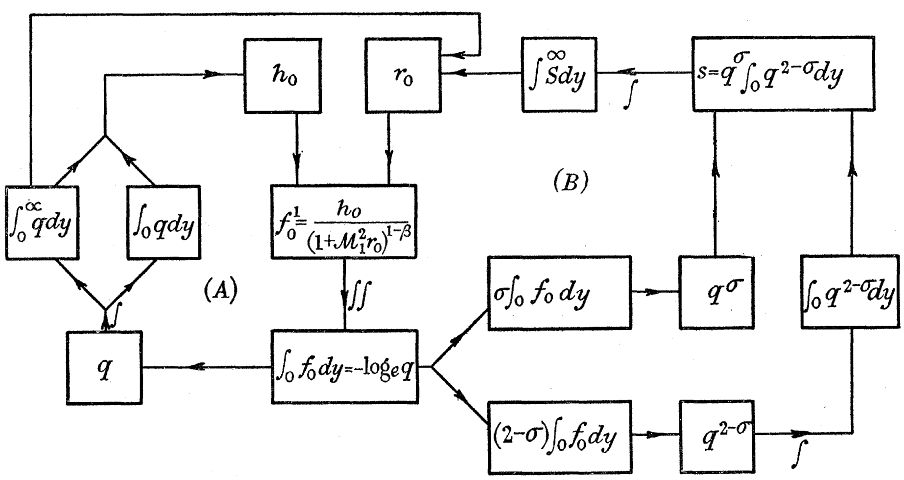

FIGURE 6. Schematic block diagram of iterative process for h_0, r_0.

The iteration is reasonably convergent, at any rate for not too large values of $\mathscr{M}_1^2$; for $\sigma = 0{\cdot}715$, $\beta = \frac{8}{9}$, $\mathscr{M}_1^2 = 10$ the improvement in the solution is by about a factor 3 per iteration.

Further, since r_0 and h_0, for given η, vary little with $\mathscr{M}_1$, it follows that once solutions have been obtained for $\mathscr{M}_1^2 = 0$ and one non-zero value, linear interpolation in $\mathscr{M}_1^2$ should give a good first estimate from which to begin the iterative process for any other value of $\mathscr{M}_1$.

The small variation of r_0 and h_0 with $\mathscr{M}_1^2$ for constant is an important advantage of η over η^* as independent variable.

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13. Evaluation of solutions of the equations for the higher-order functions

There are several possible methods by which the evaluation of solutions of the equations (8·23), (9·1), (9·2) for the higher-order functions, with the two-point boundary conditions at $\eta = 0$ and $\eta = \infty$, might be attempted.

Since these equations are linear, one possible method is to evaluate a particular integral and two complementary functions, all satisfying the conditions at $\eta = 0$, and to form such a linear combination as to fit the conditions at $\eta = \infty$. In numerical work, such a treatment of linear equations with two-point boundary conditions is often more a formal possibility than a practically useful method, but in the present case it appears practicable for numerical work, at least for the smaller values of the order n. For larger values of n the complementary functions may increase too rapidly for large η for this process to be convenient for numerical work.

For the particular integral, the values of $h_n'(0)$ and $r_n(0)$ are undetermined by the conditions at $\eta = 0$ and are free to be chosen according to the best available estimates of the values required to obtain a solution satisfying the conditions at $\eta = \infty$. It would probably be desirable to evaluate some rough trial solutions first with trial values of $h_n'(0)$ and $r_n(0)$ in order to get an idea of the values required, before evaluating an accurate particular integral from which the final solution would be determined by superposition of the complementary functions. This would avoid the situation, which otherwise would be liable to arise, that the final solution for large η was determined as the small difference of a large particular integral and a large contribution from the complementary function. It would probably be best to evaluate the complementary functions, one for $h_n'(0) = 1$, $r_n(0) = 0$ and the other for $h_n'(0) = 0$, $r_n(0) = 1$.

Various forms of iterative treatment of equations (8·23), (9·1) and (9·2) can be devised. They may be more useful in improving approximate solutions already obtained by other methods, but it is possible that an interative method may be useful even in the first rough determination of a solution satisfying the two-point boundary conditions. One iterative method, suggested by the possibility of using the differential analyzer for the solution of these equations, has been found practicable, if rather long, for numerical work. The general plan of this method is somewhat similar to that considered for the null-order equations, though the integrations cannot be reduced to quadratures.

Simultaneous solution of equations (8·23), (9·1) and (9·2) is certainly beyond the capacity of an 8-integrator differential analyzer, which is the largest at present available in Britain. On the other hand, equations (9·1) and (9·2) are of the same form, and could be handled *alternately* with only minor changes of machine set-up; the evaluation of f_n from solutions of these equations is a matter of quadrature which can be done numerically. This suggests the following procedure. First estimate h_n and r_n. Adopting these estimates, evaluate f_n' from (8·23) and integrate, giving an approximation to f_n. Then with this f_n solve equation (8·1) for h_n, and with the h_n' derived from this solution and with the same f_n, solve (9·2) for r_n. Repeat the whole process with the h_n and r_n so obtained.

With f_n regarded as given, the terms $(2n+1) f_n h_0'$ in (9·1) can be treated as a contribution to the inhomogeneous term in this equation. The evaluation of a solution with the specified boundary values of $h_n(0)$ and $h_n(\infty)$ involves the evaluation of a particular integral and a complementary function, the latter being a solution of

$$h''_n + f_0 h'_n - 2nf'_0 h_n = 0, \tag{13·1}$$

with $h_n(0) = 0$; this only involves the functions f_0, f'_0 and the value of n, so the same complementary function applies to all stages of the iterative process. The final solution is obtained by adding to the particular integral such a multiple of the complementary function that the linear combination satisfies the condition at infinity.

Similarly, with h'_n obtained from the solution of (9·1), and f_n also regarded as given, the terms $(2n+1)f_n h'_0$ and $\frac{1}{2}(\gamma-1)\,h'_0 h'_n$ in equation (9·2) can be treated as contributions to the inhomogeneous term in this equation, and a similar treatment applied. The equation for the complementary function in this treatment is

$$(1/\sigma)\,r''_n + f_0 r'_n - 2nf'_0 r_n = 0, \tag{13·2}$$

with $r'_n(0) = 0$.

This method has been tried in a numerical treatment of the equations for $n = 1$, $\mathscr{M}_1^2 = 10$, and has been found to converge satisfactorily; the improvement is by a factor of the order 3 per iteration, as with the method suggested in § 12 for the null-order functions.

For numerical integration, it is very convenient to transform a second-order differential equation into such a form that the first derivative of the dependent variable is absent. In the case of equation (9·1) this can be done by using

$$h_n^* = \{h_n - h_n(\infty)\} \exp\left\{\frac{1}{2}\int_0 f_0\,d\eta\right\} = \{h_n - h_n(\infty)\}/q^{\frac{1}{2}}, \tag{13·3}$$

as dependent variable; the equation then becomes

$$\frac{d^2 h_n^*}{d\eta^2} = \{(2n+\tfrac{1}{2})f'_0 + \tfrac{1}{4}f_0^2\}h_n^* - \{D_n - 2nf'_0 h_n(\infty) + (2n+1)f_n h'_0\}/q^{\frac{1}{2}}, \tag{13·4}$$

with boundary conditions $h_n^*(0) = -h_n(\infty)$, $h_n^*(\infty) = 0$.

Provided $\sigma > \frac{1}{2}$, the inhomogeneous term in this equation tends to zero as $\eta \to \infty$. The integration process for the equation in this form is simple, but the whole process of obtaining a solution satisfying the boundary condition is not quite straightforward, because of the very rapid increase of the complementary function for large η (it is $O(\eta^{2n} e^{\frac{1}{2}\eta^2})$), so that different particular integrals diverge very rapidly from one another. This makes it impracticable to evaluate just a complementary function and a single particular integral (unless the choice of initial value of $dh^*/d\eta$ has been exceptionally fortunate). It is necessary to evaluate one particular integral out to some value, say η_2, of η, estimate from the behaviour of this solution what multiple of the complementary function has to be added to get the required solution, and start the evaluation of another particular integral from some value η, less than η_2, and perhaps repeat this process two or three times. But the simplification introduced in the integration procedure by the absence of the first derivative more than compensates for this additional complication.

Similarly, for the equation for r_n, the use of

$$r_n^* = \{r_n - r_n(\infty)\} \exp\left\{\tfrac{1}{2}\sigma\int_0 f_0\,d\eta\right\} = \{r_n - r_n(\infty)\}/q^{\frac{1}{2}\sigma}, \tag{13·5}$$

eliminates the first derivative, and gives

$$\frac{1}{\sigma}\frac{d^2r_n^*}{d\eta^2} = \{(2n+\tfrac{1}{2})f_0'+\tfrac{1}{4}\sigma f_0^2\}r_n^* - \{E_n - 2nf_0'\,r_n(\infty) + (2n+1)f_n r_0' + \tfrac{1}{4}(\gamma-1)\,h_0'h_n'\}/q^{\frac{1}{2}\sigma}, \quad (13\cdot6)$$

with boundary conditions $dr_n^*/d\eta = 0$ at $\eta = 0$, $r_n^* = 0$ at $\eta = \infty$. The treatment of this equation is similar to that of equation (13·4).

A block diagram of this iterative process, using the equations in the form (13·4), (13·6), is shown in figure 7.

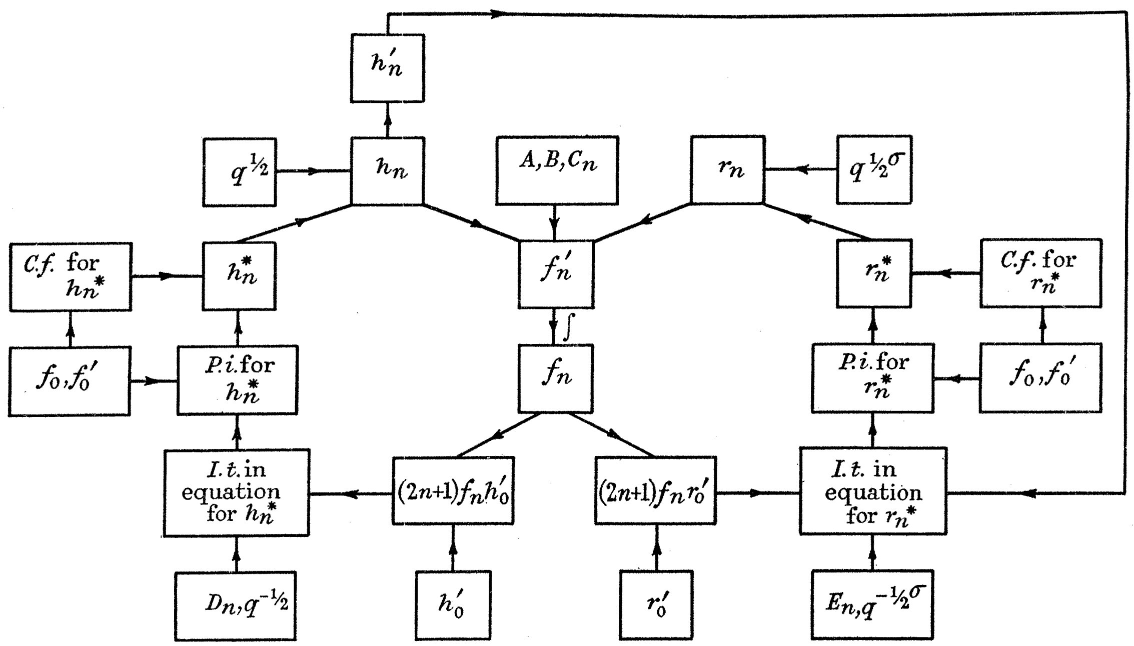

Figure 7. Schematic block diagram of iterative process for h_n, r_n $(n>0)$. *I.t.* = inhomogeneous term, *P.i.* = particular integral, *C.f.* = complementary function.

14. Means of carrying out the evaluation of solutions of the equations for the null-order and higher-order functions

Any method of evaluating solutions of the equations for several orders of functions, even for a single value of $\mathscr{M}_1^2$, will involve a substantial amount of computing work.

Since the null-order functions occur as coefficients in the equations for all higher-order functions, and for $n>4$ they do so multiplied by factors 10 or greater, and since also in the inhomogeneous terms in these equations the lower-order functions also occur multiplied by similar factors, it seems desirable to evaluate certainly the null-order functions and a few of the low-order functions (perhaps $n = 1, 2, 3$) to greater accuracy than is obtainable with the differential analyzer, though this instrument may be valuable in obtaining first approximations to the solutions, and particularly in locating the neighbourhood of the values of $h_n'(0)$, $r_n(0)$ to give solutions satisfying the boundary conditions at infinity.

The situation is somewhat different for the null-order functions on the one hand and for the functions of order $n \geqslant 1$ on the other; first, because of the different character of the equations in these two cases, non-linear for $n = 0$ but linear for $n \geqslant 1$, and secondly, and more important, because of the small variation of the null-order function with $\mathscr{M}_1^2$ (see § 17) which makes it easy to estimate good approximations from which to start an iterative calculation, and the large variation (and as yet hardly known behaviour) of the higher-order functions with $\mathscr{M}_1^2$.

Some solutions of the equations for the null-order functions were evaluated by the iterative method of § 12. It was found that this method converged satisfactorily, at least for values of $\mathscr{M}_1^2$ up to about 12, and that it was quite easy to use an accuracy adequate to give 4 figures in the results.

Subsequently, an opportunity arose of using the ENIAC, an electronic calculating machine using the principle of counting electrical pulses by means of electronic counting circuits, for this work and of exploring the possibilities of the application of this machine to the evaluation of the higher-order functions which would be much more laborious to calculate by normal computing methods. This work forms the subject of the following sections.

PART III. NUMERICAL SOLUTION OF THE EQUATIONS

15. The ENIAC

The ENIAC is a large calculating machine, which has been developed at the Moore School of Electrical Engineering of the University of Pennsylvania, Philadelphia, U.S.A., for the Ballistics Research Laboratory at Aberdeen Proving Gound, Aberdeen, Maryland. It was devised by Dr J. P. Eckert and Dr J. W. Mauchly and was developed primarily for application to the step-by-step numerical integration of the equations of external ballistics, but its organization is flexible enough for it to be used for many other extended numerical calculations. An account of it has been published elsewhere (Goldstine H. H. and A. 1946; Hartree 1946), and only a brief account of its general construction will be given here to explain the references in §§ 16, 17 to its application to the present problem.

It is essentially an automatic, high-speed, multi-register calculating machine, with facilities for addition, multiplication, division, and extraction of a square root, for accepting data from and furnishing results to the outside world, and for controlling automatically the sequence of such operations. It consists of a number of units which can be interconnected in different ways so that the whole assembly of units carries out different sequences of calculations. These interconnexions can be carried out through two sets of lines, one of which ('digit lines') transfers numerical information in the form of groups of pulses from one unit to another and the other ('program lines') transfers single pulses ('program pulses') which control the sequence of operations of the various units. The units are also permanently connected through a set of lines to a pulse generator which transmits a standard pattern of pulses on these lines every 200 μsec.; this, being the time required for an addition, is called an 'addition time' and is the natural unit of time in which to assess the time taken by other operations. Individual pulses in a numerical group are spaces at 10 μsec. interval.

The main units of the machine are 20 'accumulators' each of which is both an adding unit and a register and has a capacity of 10 figures with a sign indication. Subtraction is carried out by addition of the complement, multiplication by small integers is carried out by repeated addition. Multiplication in general is carried out by a high-speed multiplying unit which gives results considerably more quickly than they would be obtained by continued addition. There are three function tables, in each of which 104 values of any function required in the computation can be preset by hand-operated switches. There is a 'constant transmitter' which can store on relay registers eight numbers read from a punched card by a card reader, and a card punch by which results of the computation can be punched so as to form a permanent record. A multiplication takes a time depending on the number of digits in the multiplier, but not longer than 14 addition times or 2·8 msec.; feeding and reading a punched card and punching results on a card take about half a second each, or some thousands of addition times. Since these are comparatively slow processes, it is desirable to arrange the work so as to avoid extensive use of them, if possible.

A unit is normally quiescent. An instruction to operate consists of a pulse on one of the program lines to which it is connected, and what the unit does when stimulated by such a pulse is controlled by the positions of switches which are set by hand before the computation is started; the setting of these switches and the way the units are connected to the digit lines and program lines form the 'set-up' of the machine for any particular calculation. In any operation at least two units are concerned, one or more to transmit and one or more to receive; on completion of an operation, one of the units concerned transmits a pulse to a program line, and this then stimulates the units involved in the next operation of the computing sequence.

The card reader and card punch are the units by which the machine respectively obtains numerical information from, and supplies results to, the outside world. Information about the computing sequences involved in the problem to be handled is furnished by the interconnexions set up between the different units.

A most important unit in the organization of a computation as a whole is that called the 'master programmer', by means of which a computing sequence may be repeated, or changed automatically either at pre-determined stages of the work or at stages determined by one or more criteria evaluated in the course of the computation itself. This unit operates by switching program pulses from one line to another. It consists of ten separate six-position electronic switches called 'steppers' with each of which a counter can be associated. Each stepper has four input channels, of which three are involved in the present work, namely 'normal', 'stepper direct' and 'stepper clear' inputs; it also has one output channel for each switch position. A pulse input on the 'normal input' channel is counted by the counter and gives rise to an output pulse on a channel corresponding to the switch position, so that by connecting two or more of these output channels to different program lines from which different computing sequences are initiated, selection between these sequences is effected by control of the switch position of the stepper. Repetition of a computing sequence is effected by closing the ring of computing operations through a stepper; so long as this remains in the same switch position, the sequence will be repeated, but this repetition is broken if the stepper moves to another position.

A stepper can be stepped from one position to another in two ways, either by the receipt of a specified number of pulses applied to its normal input, this number being fixed by the settings of a group of switches which are set by hand before the computation is started, or by a pulse applied to the 'stepper direct' input; the latter does not give rise to the emission of a program pulse. The first method is used when it is desired to change the computing sequence after a pre-determined number of repetitions, the second when it is required to change it according to some criterion evaluated in the course of the computation. This criterion can usually be arranged to be that the sign of some number should be negative; then when this sign is positive, no pulse is emitted on the line connected to the stepper direct input, and the position of the stepper remains unchanged; but when the sign is negative, a pulse is emitted on this line, and the stepper moves to its next position. A stepper can be cleared back to its first position in two ways, either after moving through any specified number of other positions, or by a pulse applied to the 'stepper clear' input.

The number of steppers available, and the possibility of interconnexions between them, impart a very considerable degree of flexibility to the ENIAC in its application to intricate computations.

A simple example of the use of the master programmer is provided by its use in the evaluation of the $-(\frac{1}{9})$th power of a number, a process involved in the evaluation of solutions of the null-order functions for the value $\beta = \frac{8}{9}$ adopted; a similar process would apply for any integral value of $1/(1-\beta)$. This will be considered in some detail as an illustration of the ideas used in applying the master programmer to the organization of the whole calculation.

The notation used for connexions of a stepper is shown in figure 8; the three circles labelled *Sd*, *n*, *Sc* represent the stepper direct, normal, and stepper clear input terminals; each rectangle such as *PQRS* represents a single switch position of the stepper, the vertical line below it represents the program line to which the output from this switch position is connected, and the number N in the rectangle gives the number of pulses on the normal input channel after which the stepper steps to its next switch position. Each stepper has a reference letter (A in figure 8), and the number (n in figure 8) after it indicates the number of operative switch positions after which the stepper clears back to its first position.

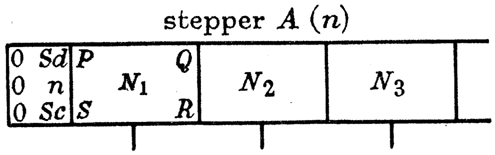

FIGURE 8. Diagrammatic representation of stepper.

The calculation of the $-(\frac{1}{9})$th power of a quantity z is done by use of the iteration formula

$$x_{n+1} = \tfrac{1}{9}x_n\{10 - zx_n^9\}, \tag{15·1}$$

and the ninth power required in the evaluation of this formula is obtained by repeated cubing:

$$x_n^9 = \{(x_n)^3\}^3. \tag{15·2}$$

With certain restrictions on x_0, $x_n \to z^{-\frac{1}{9}}$, and further this iteration formula is 'second-order', that is, at each iteration the error (if small) is squared, so that the number of correct figures is doubled. The criterion for an adequate approximation to $z^{-\frac{1}{9}}$ is taken to be

$$|x_n - x_{n-1}| < \epsilon. \tag{15·3}$$

The complete master-programmer set-up, shown diagrammatically in figure 9, involves two steppers, one (A) to control the repetition of the cubing process, and the other (B) to control the repetition of the iteration for the ninth root itself. The program lines are numbered 1, 2, ..., and the directions in which programme pulses are transmitted are indicated by arrows. The blocks labelled I, II, ..., represent diagrammatically various sequences of computing operations† initiated by programme pulses input into the units involved in the first operation of each sequence.

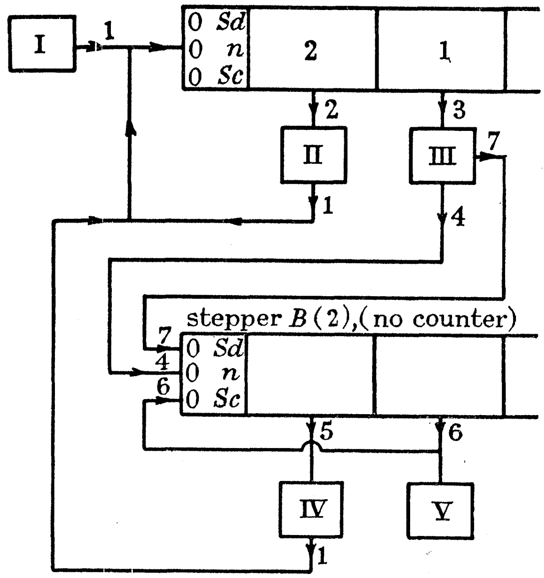

FIGURE 9. Master-programmer connexion for iteration for $-(\frac{1}{9})$th power. For explanation of symbols I to V see text.

In specifying these computing sequences, K, L, M, ... will be used to indicate different accumulators and k, l, m, ... the numbers registered by them; different contents of an accumulator which may be registered at different times will be distinguished by suffixes (but the numerical order of the suffixes does not necessarily indicate the time-sequence of the contents of an accumulator). These sequences are as follows:

(I) Transfer z to accumulator K; obtain a first estimate x_0 to $z^{-\frac{1}{9}}$, and transfer to accumulators L, M; i.e. make $k_1 = z$, $l_1 = m_1 = x_0$. When completed, emit a pulse on program line 1.

(II) If l_2 is the content of accumulator L, form $(l_2)^3$; clear accumulator L and transfer $(l_2)^3$ to it, i.e. make $l_3 = (l_2)^3$. When completed, emit a pulse on program line 1.

(III) If k, l_4, m_2 are contents of accumulators K, L, M, form $m_2(10-kl_4)/9$; clear accumulator L and transfer this quantity to it, i.e. make

$$l_5 = \tfrac{1}{9}m_2(10-kl_4). \tag{15·4}$$

Also form $(l_5-m_2)^2-\epsilon^2$; if this is negative, and not otherwise, emit a pulse on program line 7. When completed, and after the time at which the pulse on line 7, if any, would be emitted, emit a pulse on program line 4.

(IV) Transfer l_5 to accumulator M, holding it in accumulator L, i.e. make $m_3 = l_5$. When completed, emit a pulse on program line 1.

(V) Subsequent computation when $x = z^{-\frac{1}{9}}$ has been determined.

† Note that in figure 9 the blocks represent neither *quantities* involved in the computation as in figures 6 and 7, nor *components* of the machine, but *processes*.

The way in which x_0 is obtained will depend on the rest of the calculation of which this evaluation of $z^{-\frac{1}{9}}$ forms part. As a result of the sequence of operations I, the contents of the three accumulators involved are

$$k = z, \quad l = m = x_0. \tag{15·5}$$

The pulse emitted on program line 1 on completion of this sequence goes to the normal input of stepper A, and this, in its first position, emits a pulse on program line 2; the counter associated with it registers 1 pulse received on the normal input. The pulse emitted on program line 2 initiates the computing sequence II, which replaces the content of accumulator L by its cube. Hence on completion of this sequence, the contents of the accumulators are

$$k = z, \quad l = x_0^3, \quad m = x_0.$$

The pulse emitted on program line 1 at the completion of this sequence goes to the normal input of stepper A; this, being still in its first position, emits a pulse on program line 2 and, being set to step after receiving two pulses on its normal input in its first position (indicated by the number 2 in the first position of stepper A in figure 9) it steps to its second position. The pulse emitted on program line 2 results in the repetition of the sequence II, giving finally the results

$$k = z, \quad l = x_0^9, \quad m = x_0$$

and the emission of a pulse on program line 1.

This pulse goes to the normal input of stepper A, and this, being now in its second position, emits a pulse on program line 3; and further, being set to step after one pulse received on its normal input in its second position, and to clear to its first position on stepping from its second position (indicated by the number 2 in $A(2)$), it then clears back so as to be ready to initiate the cubing process again in the next iteration. The pulse on program line 3 initiates the computing sequence III, as a result of which, first, the content of accumulator L becomes $x_0(10-zx_0^9)/9$ which is x_1 according to the iteration formula (14·1), i.e.

$$k = z, \quad l = x_1, \quad m = x_0,$$

secondly, the sign of $(x_1-x_0)^2-\epsilon^2$ is determined. If this sign is negative, it means that the criterion (14·3) is satisfied, x_1 is already an adequate approximation to $z^{-\frac{1}{9}}$, and the subsequent calculation for which this value of $z^{-\frac{1}{9}}$ is required can be undertaken; in this case a pulse on program line 7 to the stepper direct input of stepper B steps it to its second position, and the pulse on program line 4 subsequently emitted and applied to the normal input of stepper B results in the emission of a pulse on program line 8. This both initiates the computing sequence V for the subsequent calculation and clears stepper B back to its first position through its stepper clear input; this is necessary since, in the set-up indicated in figure 9, no counter is associated with stepper B.

If on the other hand the sign of $(x_1-x_0)^2-\epsilon^2$ is positive no pulse is applied to the stepper direct input of stepper B, and this stepper remains in its first position, so that the pulse on program line 4 results in the emission of a pulse on program line 5. This initiates the sequence IV, which consists simply in the replacement of the content of accumulator M by that of L, which gives

$$k = z, \quad l = m = x_1.$$

This is just the situation (15·5) with x_0 replaced by x_1, and the pulse emitted on program line 1, going to stepper A, now cleared back to its first position and with its counter at zero, will repeat the whole process, and this repetition will continue until the criterion (15·3) is satisfied, when the operation of stepper B through its stepper direct input will break this repetition and initiate the computing sequence V for the subsequent calculations.

This is only one of several possible ways of evaluating $z^{-\frac{1}{6}}$. In the set-up for integrating the null-order equations, a somewhat different method was actually used, but it does not provide such a good illustration of the use of the master programmer. In this case x_0 was taken from a function table, and was known to be good to 1 in 500; then two iterations were certainly good enough for the use to be made of the value of $z^{-\frac{1}{6}}$ in the subsequent calculations. Stepper B then had a counter associated with it, and was set to step from its first position after 2 normal input pulses, and to clear back from its second position after one input pulse (i.e. $n = 2$, $N_1 = 2$, $N_2 = 1$ in the notation of figure 8), and the criterion (15·3) was not used.

16. Application of the ENIAC to the null-order equations

There are two main ways of trying to evaluate a solution of the non-linear null-order equations (8·24 i), (8·25 i) and (8·27 i) with the two-point boundary conditions (9·6). One is to use an iterative method such as that of § 12; the other is to evaluate solutions with estimated values of $h'(0)$, $r(0)$, and adjust these by trial until a solution satisfying the conditions at infinity has been found.

The iterative method would involve keeping a record of f_0 for the nth iteration to use as input for the $(n+1)$th iteration, and probably of two or three other functions (such as q, $\int_0 q\,d\eta$) as well; if 120 values of each were required (say covering the range $\eta = 0$ to 6 at intervals of 0·05), this would require a memory capacity for several hundred numbers in the course of each iteration, quite apart from the memory capacity required for intermediate results needed during the calculation of a single interval of the iteration. With a machine with adequate memory capacity into which numbers could be recorded and from which they could be read in times of the order of a few addition times, such a method would probably be the best, but with the ENIAC, with its immediately accessible memory limited to the twenty accumulators, such an iterative method would involve considerable use of punched cards, a deck of cards, or several decks, being punched, then transferred from card punch to card reader, then read during a subsequent stage of the calculation. The punching and reading of cards are very slow operations compared with the other operations of the ENIAC, and moreover the need for an operator to transfer the decks of cards from the punch to the reader breaks the automatic operation which is such an important feature of large computing machines such as the ENIAC.

A scheme for using this machine in this way could certainly be developed, but it seemed that a more effective approach would be made by evaluating a set of solutions with trial values of $h_0'(0)$, $r_0(0)$, and such a method was adopted.

Even if the trial values of $h_0'(0)$ and $r_0(0)$ are incorrect, h_0 and r_0 evaluated by outward integration of equations (8·25 i) and (8·27 i) will tend to *some* constant values, and it was estimated from the preliminary work with the iterative method that to 6-figure accuracy the asymptotic values would be reached by $\eta = 5$. Hence the conditions at infinity were replaced by the conditions:

$$h(5) = 2, \quad r(5) = 0, \tag{16·1}$$

and integration was carried out over the range $\eta = 0$ to 5 only except in one case when it was carried to $\eta = 8$.

To limit demands on memory capacity it is convenient in step-by-step numerical integration to use a simple integration formula, not involving information about the integrand outside the interval through which the integration is being carried; and hence, to limit the aggregate error due to the approximations of the integration formula (sometimes called 'truncation errors'), it is necessary to use a small interval of integration. With a machine of the high computing speed of the ENIAC, this is not a serious drawback. With the computing schedule used, which will be explained later, the machine evaluated solutions at the rate of about 8 intervals a second, so that with intervals of $\delta\eta = 0{\cdot}02$, taking 250 to cover the range $\eta = 0$ to 5, evaluation of a trial solution took about half a minute.

In a trial solution, only one card was punched, at the end of the solution, with the values of

$$h_0'(0), \quad r(0), \quad h(5), \quad r(5). \tag{16·2}$$

Five solutions with $(h'(0), r(0)) = (a, b)$; $(a\pm\alpha, b)$; $(a, b\pm\beta)$ then give approximate variations of $h(5)$, $r(5)$ with the estimated values of $h'(0)$, $r(0)$; three solutions would have been enough for this purpose but five were run to provide a check (the additional two only took a minute of machine time). From these variations, approximate values of $\Delta h'(0)$, $\Delta r(0)$ required to give specified values $\Delta h(5)$, $\Delta r(5)$ could be calculated, and supplied to the machine through the card reader. With this information it could then use the values of $h(5)$ and $r(5)$, obtained from one solution with trial values of $h'(0)$ and $r(0)$, to interpolate better values of $h'(0)$ and $r(0)$, and so, by alternatively evaluating a solution with trial values of $h'(0)$ and $r(0)$ and using the results to estimate better values, to arrive automatically at a solution satisfying the two-point boundary conditions to any required accuracy within its capacity. After each interpolation of values of $h'(0)$ and $r(0)$ for the next solution, the quantity

$$[10^6(h(5)-2)]^2+[10^7 r(5)]^2-5, \tag{16·3}$$

was evaluated and its sign used as a criterion as to whether the *next* solution was going to fit the conditions at $\eta = 5$ well enough to be regarded as the final one. If this sign was negative, then

$$h(5)-2<\surd 5\times 10^{-6}, \quad r(5)<\surd 5\times 10^{-7},$$

for the trial solution just completed, and with the better values of $h'(0)$, $r(0)$ just evaluated, the next solution would certainly be better than this and should not depart from the conditions (16·1) by more than $1{\cdot}10^{-6}$ in either $h(5)$ or $10r(5)$. This was regarded as adequate, and the negative sign of (16·3) was used to operate steppers on the master programmer with connexions so arranged that in the next and final solution the machine punched a card, giving h', h, r', r, f' and f, for *each* interval of the integration, instead of one card at the end of the integration only.

The switching of the computing sequence from step-by-step integration to selection of new initial conditions, of this selection by reading a card or by interpolation from the results of the previous solution, and of the punching from taking place at $\eta = 5$ only to taking place at each step of the integration, was all set up by connexions to and from the master programmer, and carried out automatically in the operation of the machine, so that once the first estimate of $(h'(0), r(0))$ and the values of

$$\frac{\Delta h'(0)}{\Delta h(5)}, \quad \frac{\Delta h'(0)}{\Delta r(5)}, \quad \frac{\Delta r(0)}{\Delta h(5)}, \quad \frac{\Delta r(0)}{\Delta r(5)} \tag{16·4}$$

had been supplied to the machine by means of a card, the whole of the rest of the calculation, involving the simultaneous determination of the two independent parameters $h'(0)$, $r(0)$ so as to satisfy the two conditions at infinity, and the evaluation and tabulation of the corresponding solution was carried out without the further attention of an operator. The evaluation of a single trial solution took about $\frac{1}{2}$ min. as already mentioned; the evaluation of the final solution, involving punching a card for every interval of the integration, took about $2\frac{1}{2}$ min. The total time depended on how many trial solutions had to be made, but was about 4 min. for each value of $\mathscr{M}_1^2$.

The full procedure outlined above after (16·2) was carried out for $\mathscr{M}_1^2 = 10$ only. For the other values of $\mathscr{M}_1^2$ it was shortened as follows: $\dfrac{\Delta h(5)}{\Delta h'(0)}, \dfrac{\Delta r(5)}{\Delta h'(0)}$ and $\dfrac{\Delta r(5)}{\Delta r(0)}$ were taken the same for each $\mathscr{M}_1^2$ as for $\mathscr{M}_1^2 = 10$, and $\dfrac{\Delta h(5)}{\Delta r(0)}$ was taken as proportional to $\mathscr{M}_1^2$ (it is certainly 0 for $\mathscr{M}_1^2 = 0$), and on this basis the quantities (15·4) were evaluated for $\mathscr{M}_1^2 = 0$ and 20 and interpolated linearly in $\mathscr{M}_1^2$ for other values. This is certainly not accurate, but even if slightly incorrect values of the quantities (16·4) are used the only result is that a few more trial solutions have to be made before the criterion given by the sign of (16·3) is satisfied, and even if six trials have to be made instead of two, this only adds 2 min. to the total time of obtaining a final solution, whereas evaluating the quantities (16·4) from three (or five) preliminary solutions for each value of $\mathscr{M}_1^2$, and punching and verifying a card for supplying these data to the machine would take longer than this, and the alternative process of working out an addition to the machine set-up to evaluate the quantities (16·4) from such a set of solutions would take longer than the aggregate time taken by the few additional solutions needed on account of the use of inaccurate values of the quantities (16·4). So the apparently crude use of approximate values of the quantities (16·4) is in fact the most efficient. This, of course, is a result of the high computing speed of the ENIAC and the short time which it takes to evaluate a single trial solution.

An example of the results of this process is shown in table 1. The first two columns give the successive estimates of $h_0'(0)$ and $r_0(0)$, the second two the resulting values of $h_0(5)$ and $r_0(0)$.

TABLE 1. EXAMPLE OF SUCCESSIVE APPROXIMATIONS TO $h_0'(0)$, $r_0(0)$. $\mathscr{M}_1^2 = 2$

	initial values		final values	
	$2h_0'(0)$	$2r_0(0)$	$h_0(5)$	$2r_0(5)$
first trial	2·60100000	0·33745000	2·00043562	−0·00018755
second trial	2·60013708	0·33749239	2·00000212	+0·00000108
third trial	2·60013353	0·33749054	2·00000009	−0·00000022
final solution	2·60013323	0·33749072	2·00000001	−0·00000006

The master-programmer set-up for this overall control of the computations is shown in figure 10. In view of the full discussion of the example in § 14, it is hoped that this more complex set-up will be able to be followed with only a brief explanation.

Stepper C controls the selection of initial conditions. In the full procedure outlined after (16·2), the switch controlling the number of repeats in the first switch position is set to 5, and the output connexion from the second switch position is broken at X, so that after completion of the fifth integration, the program pulse on program line 1, input into stepper C in its second position, results in no program pulse being output to any unit of

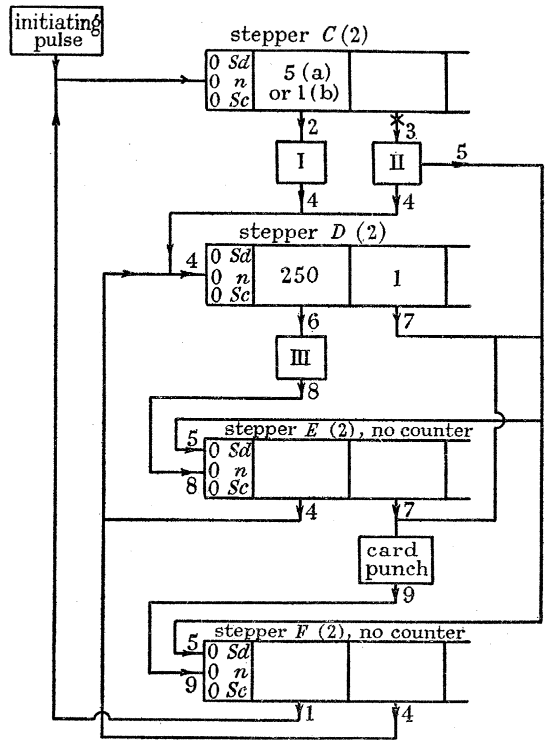

FIGURE 10. Master-programmer set-up for control of computation of null-order functions. Computing sequences:

(I) (a) Clear all accumulators. Feed card and read $H(0)$, $r(0)$ from card or
(b) Clear all accumulators. Feed card and read $H(0)$, $r(0)$, also

$$\frac{\Delta H(0)}{\Delta h(5)},\quad \frac{\Delta r(0)}{\Delta h(5)},\quad \frac{\Delta H(0)}{\Delta r(5)},\quad \frac{\Delta r(0)}{\Delta r(5)}.$$

When completed, emit a pulse on program line 4. [$H = h'_0\,\delta\eta$; see (17·1) below.]

(II) Interpolate for new initial conditions. Evaluate (15·3) and if the sign is negative, and not otherwise, emit a pulse on program line 5. When completed, emit a pulse on program line 4.

(III) Computing sequence for one interval of the integration.

the machine, which then stops. In this case each of the five cards to be read contains values of $h'(0)$ and $r(0)$ only. When the values of the quantities (16·4) are available, the card to be read contains these quantities as well as $h'(0)$ and $r(0)$, and the repeat switch is set to 1, so that after one integration with these values of $h'(0)$ and $r(0)$, a program pulse on line 1 results in the interpolation procedure for values of $h'(0)$ and $r(0)$ for the next solution, instead of these being taken from a card.

Stepper D controls the number of integration intervals evaluated in a trial solution before the values of the quantities (16·2) are punched. Steppers E and F are concerned with the control of the punch, results (16·2) only being punched until the criterion given by the sign

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of (16·4) is satisfied, after which the punching of results is to be part of the sequence of operations for each interval of integration. The punch has only one input terminal and one output terminal for program pulses, and two steppers have to be used in conjunction with it, one for switching the punch into the computing procedure for an interval of the integration, and the other for controlling the operation resulting from the output program pulse from the punch.

17. The computing procedure for a single interval of the integration

As already stated, to limit demands on memory capacity it is advisable, in using the ENIAC for numerical integration of differential equations, to use an integration formula which does not involve reference to the behaviour of the integrands outside the interval through which the integration is being carried. In general, in order to obtain such a formula with errors of higher order than the second in the interval length $\delta\eta$, it is necessary either to differentiate the equations in order to have values for some higher derivatives to use in a Taylor series from the beginning of the interval, or to use some method involving either estimation or successive approximation for some quantity at the end of the interval.

But in the present case the form of the equations is such that results correct to second order in $\delta\eta$ can be obtained without any of these processes, if the equations are taken in the right order. The features of the equations which make this possible are:

(i) The equation (8·24 i) giving f_0' involves h_0 and r_0 only,

(ii) The equation (8·25 i) giving h_0'' is linear in h_0' and does not involve r_0',

(iii) The equation (8·27 i) giving r_0'' is linear in r_0'.

In the following discussion of the integration formulae, the suffixes 0 indicating the null-order of the functions concerned will be omitted as no confusion can arise; suffixes (0), (1) in brackets will be used to denote quantities at the beginning and end of an interval of the integration. Also H and R will be used for the reduced derivatives.

$$H = h_0'\,\delta\eta, \quad R = r_0'\,\delta\eta \tag{17·1}$$

and it is convenient also to write

$$F = \tfrac{1}{2} f_0\,\delta\eta. \tag{17·2}$$

It will be seen later that h_0'' and r_0'' are never evaluated, so that the quantities known at the beginning of an interval are

$$f'_{(0)}, \quad F_{(0)}, \quad H_{(0)}, \quad h_{(0)}, \quad R_{(0)}, \quad r_{(0)} \tag{17·3}$$

which are the corresponding quantities with suffix (1) for the previous interval.

For a single integration the two main formulae with third-order errors are

$$z_{(1)} - z_{(0)} = z'_{(0)}\,\delta\eta + \tfrac{1}{2} z''_{(0)} (\delta\eta)^2 + O(\delta\eta)^3 \tag{17·4}$$

and

$$z_{(1)} - z_{(0)} = \tfrac{1}{2}\{z'_{(0)} + z'_{(1)}\}\,(\delta\eta) + O(\delta\eta)^3. \tag{17·5}$$

Since h_0 and r_0 are given by second-order equations, (17·4) can be used to give $h_{(1)}$ and $r_{(1)}$ correct to $O(\delta\eta)^2$ by putting $z = h_0$, r_0 and substituting for h_0'', r_0'' in the $(\delta\eta)^2$ term from (8·25 i), (8·27 i) respectively. This gives, with errors $O(\delta\eta)^3$,

$$h_{(1)} - h_{(0)} = \{1 - F_{(0)}\}\,H_{(0)} \tag{17·6}$$

and

$$r_{(1)} - r_{(0)} = \{1 - \sigma F_{(0)}\}\,R_{(0)} - \tfrac{1}{2}\sigma(\gamma - 1)\,\{H_{(0)}\}^2 \tag{17·7}$$

From $h_{(1)}$ and $r_{(1)}, f'_{(1)}$ can be calculated from (8·24 i) and so $f_{(1)}$ from (17·5), with $z = f_0$; or in terms of F

$$F_{(1)} - F_{(0)} = (\tfrac{1}{2}\delta\eta)^2 \{f'_{(1)} + f'_{(0)}\}. \tag{17·8}$$

Putting $z = H$ in (17·5) and substituting for h''_0 gives

$$H_{(1)} - H_{(0)} = F_{(1)} H_{(1)} - F_{(0)} H_{(0)}$$

or

$$H_{(1)} = \frac{1 - F_{(0)}}{1 + F_{(1)}} H_{(0)} \tag{17·9}$$

correct to $O(\delta\eta)^3$ (giving h'_0 correct to $O(\delta\eta)^2$), and similarly

$$R_{(1)} = \frac{1 - \sigma F_{(0)}}{1 + \sigma F_{(1)}} R_{(0)} - \frac{\sigma}{1 + \sigma F_{(1)}} \frac{\gamma - 1}{2} [\{H_{(1)}\}^2 + \{H_{(0)}\}^2] \tag{17·10}$$

to the same order in $(\delta\eta)$.

When $H_{(1)}$ and $R_{(1)}$ have been found from (17·9) and (17·10) it would be possible to recalculate $h_{(1)}$ and $r_{(1)}$ from the formulae derived from (16·5), namely

$$\left.\begin{aligned} h_{(1)} - h_{(0)} &= \tfrac{1}{2}(H_{(1)} + H_{(0)}), \\ r_{(1)} - r_{(0)} &= \tfrac{1}{2}(R_{(1)} + R_{(0)}). \end{aligned}\right\} \tag{17·11}$$

These have third-order errors as (17·6), (17·7) have, but the magnitude of the third-order errors of (17·11) are half those of (17·5), (17·6), and it might be as well to use (17·11) to reduce the accumulated second-order error, although the values of $h_{(1)}$, $r_{(1)}$, calculated from (17·6), (17·7) would be adequate for evaluating $f'_{(0)}$, and similarly in the other formulae. The $O(\delta\eta)^2$ errors can, if appreciable, be estimated by Richardson's (1927) process of 'h^2-extrapolation'. Some results are given in § 19.

The computing sequence used for an interval of integration on the ENIAC was based on formulae (17·6) to (17·10). The evaluation of $(1 + \mathscr{M}_1^2 r_{(0)})^{-\frac{1}{9}}$ required in the evaluation of $f'_{(1)}$ from $h_{(1)}$ and $r_{(1)}$ was carried out as explained in § 15. It will be seen that this involved use of the master programmer in one small component of the main computing sequence as well as in the main organization of the computation as a whole, and moreover independently of this.

18. Application of the ENIAC to the equations for the higher-order functions ($n \geqslant 1$)

As for the null-order equations, the solution of the higher-order equations can be carried out either by some iterative method or by integration from $\eta = 0$ with trial values of $h'_n(0)$, $r_n(0)$, these values being adjusted so that the solution satisfies the conditions at infinity, and, for the same reasons as those given in § 16, the latter is the most suitable for application of the ENIAC.

In the solution of these equations by a method of this kind, there are four main types of computing routine involved.

(*a*) Evaluation of the input functions

$$f_0, \quad f'_0, \quad h'_0, \quad r'_0, \quad 1/(1 + \mathscr{M}_1^2 r_0)^{1-\beta}, \quad (1-\beta)\mathscr{M}_1^2 h_0/(1 + \mathscr{M}_1^2 r_0),$$

which occur in the coefficients of the functions to be determined in *all* the higher-order equations (f'_0 with a factor $2n$ and h'_0, r'_0 with a factor $(2n+1)$);

(*b*) evaluation of the inhomogeneous terms C_n, D_n, E_n given by (9·3), (9·4), (9·5), which are peculiar to each value of n;

(*c*) procedure for a single interval of the integration;

(*d*) determination of $h'_n(0)$, $r_n(0)$ to satisfy the boundary conditions at infinity, and evaluation of the corresponding solution.

Of these, (*a*) and (*b*) cannot usefully be considered until it is known in what form the data will be most conveniently expressed for use in (*c*), and it will be convenient to consider this first. It will be understood that the sets of equations are solved in order of n increasing, so that in the solution of the equations for the function of any order n, the functions of order $j<n$ are known.

The equations (8·23), (9·1), (9·2) have properties similar to the three properties (§ 17 (i) to (iii)) of the equations for the null-order functions, namely:

(i) the equation for f'_n involves only h_n, r_n and known functions;

(ii) the equation for h''_n is linear in h'_n and does not involve r'_n;

(iii) the equation for r''_n is linear in r'_n.

The fact that the system of equations as a whole is linear in f_n, r_n, h_n is *not* important at this stage. These three properties enable a procedure very like that of § 17 to be used, with similar consequences, that the aggregate errors of the results are $O(\delta\eta)^2$ and that this accuracy is obtained without successive approximation or estimation.

If, to avoid the use of suffixes both to indicate order of function and to distinguish between values at the beginning and end of an interval, we write

$$h_n = u, \quad h'_n\,\delta\eta = U, \quad r_n = v, \quad r'_n\,\delta\eta = V, \quad f'_n = \psi, \quad \tfrac{1}{2}f_n\,\delta\eta = \Psi$$

and also for shortness write

$$P = \tfrac{1}{2}(\delta\eta)^2\,(h''_n + f_0 h'_n) = n(\delta\eta)^2 f'_0\, u - (2n+1)\,\Psi H - \tfrac{1}{2}(\delta\eta)^2 D_n,$$

$$Q = \tfrac{1}{2}(\delta\eta)^2\left(\frac{1}{\sigma}r''_n + f_0 r'_n\right) = n(\delta\eta)^2 f'_0\, v - (2n+1)\,\Psi R - \tfrac{1}{4}(\gamma-1)\,HU - \tfrac{1}{2}(\delta\eta)^2 E_n,$$

the equations are

$$u_{(1)} - u_{(0)} = \{(1 - F_{(0)})\}\,U_{(0)} + F_{(0)}, \tag{18·1}$$

$$v_{(1)} - v_{(0)} = \{(1 - \sigma F_{(0)})\}\,V_{(0)} + \sigma Q_{(0)}, \tag{18·2}$$

$$\psi' = A(u - Bv) + C, \tag{18·3}$$

$$\Psi_{(1)} - \Psi_{(0)} = (\tfrac{1}{2}\delta\eta)^2\,(\psi'_{(1)} + \psi'_{(0)}), \tag{18·4}$$

$$U_{(1)} = \frac{1 - F_{(0)}}{1 + F_{(1)}}\,U_{(0)} + \frac{1}{1 + F_{(1)}}\,\{(P_{(1)} + P_{(0)}\}, \tag{18·5}$$

$$V_{(1)} = \frac{1 - \sigma F_{(0)}}{1 + \sigma F_{(1)}}\,V_{(1)} + \frac{\sigma}{1 + \sigma_{(1)}}\,\{Q_{(1)} + Q_{(0)}\}, \tag{18·6}$$

$$u_{(1)} - u_{(0)} = \tfrac{1}{2}\{U_{(1)} + U_{(0)}\}, \tag{18·7}$$

$$v_{(1)} - v_{(0)} = \tfrac{1}{2}\{V_{(1)} + V_{(0)}\}. \tag{18·8}$$

Equations (18·1), (18·2) were used for evaluating the $u_{(1)}$ and $v_{(1)}$ for use in calculating $\psi_{(1)}$, the values derived from (18·7) and (18·8) being carried over as $u_{(0)}$, $v_{(0)}$ for the next interval of the integration. This was done partly for the reason given after (17·11), and partly because, for reasons connected with the use of cards for supplying numerical data to the machine, some accuracy in the values of $F_{(0)}$ had to be sacrificed; enough was retained for the calculation of ψ', but it seemed advisable to obtain $u_{(1)}$ and $v_{(1)}$ from the more accurate formulae (18·7), (18·8), especially as these formulae are very simple and quick to evaluate.

These show that it would be best to evaluate once for all the quantities

$$\frac{1-F_{(0)}}{1+F_{(1)}}, \quad \frac{1}{1+F_{(1)}}, \quad \frac{1-\sigma F_{(0)}}{1+\sigma F_{(1)}}, \quad \frac{\sigma}{1+\sigma F_{(1)}},$$

for each interval of integration, and so eliminate the comparatively slow process of division from the evaluation of each solution; other data to be supplied are

$$F_{(0)}, \quad H_{(1)}, \quad R_{(1)}, \quad A = 1/(1+\mathscr{M}_1^2 r_0)^{1-\beta}, \quad B = (1-\beta)\,\mathscr{M}_1^2 h_0/(1+\mathscr{M}_1^2 r_0),$$

and the values of the inhomogeneous terms

$$C_n, \quad D_n, \quad E_n,$$

for each interval of the integration; this is, altogether, 250 values of each of 12 functions. This is much more than the capacity of the three function tables and shows that it is quite necessary to use punched cards. Although use of these slows down the overall computing speed of the machine considerably it leaves it still fast compared with other means of computation; the evaluation of the 250 intervals of a solution might take 5 or 10 min. instead of $\frac{1}{2}$ min., but this is still quite fast.

Twelve functions to better than 5-figure accuracy is beyond the capacity of the relay register of the 'constant transmitter' of the ENIAC. Further some of the data is common to the equations for the functions of all orders n, whereas the rest is peculiar to each particular value of n, and it is desirable to keep these two kinds of information on different cards. So it is necessary to use more than one card per interval, and, to limit demands on the memory capacity of the accumulators, it seemed best to use three cards per interval, one with the values of

$$F_{(0)}, \quad H_{(1)}, \quad R_{(1)}, \quad A, \quad B, \tag{18·9}$$

which are independent of n and are wanted early in the calculation for that interval, one with the values of

$$(C_n)_{(1)}, \quad (D_n)_{(1)}, \quad (E_n)_{(1)}, \tag{18·10}$$

and one with the values of

$$\frac{1-F_{(0)}}{1+F_{(0)}}, \quad \frac{1}{1+F_{(1)}}, \quad \frac{1-\sigma F_{(1)}}{1+\sigma F_{(1)}}, \quad \frac{\sigma}{1+\sigma F_{(1)}}, \quad H_{(1)}, \quad R_{(1)}, \tag{18·11}$$

which are independent of n but are wanted later in the interval, in particular after $(C_n)_{(1)}$ and $(D_n)_{(1)}$ which must be taken from the card of that deck which is peculiar to each value of n.

The decks of cards with the data (18·9) and (18·11) which apply to all values of n will be called 'Deck 1' and 'Deck 3' respectively; that with the data (18·10) appropriate to the equations for the functions of any particular order n will be called 'Deck $(2, n)$'. For $n = 1$ it seemed more convenient to evaluate

$$C_1^* = C_1(1+\mathscr{M}_1^2 r_0)^{1-\beta} = \beta m h_0, \tag{18·12}$$

and to use equation (8·24 ii) in the form

$$f_1' = A(h_1 - Bv_1 - C_1^*),$$

the values of D_1 and E_1 are

$$D_1 = \phi_0 = (1 + \mathscr{M}_1^2 r_0)^{1-\beta}, \quad E_1 = \tfrac{1}{4}\phi_0 h_0; \tag{18·13}$$

deck (2·1) was punched with the values of

$$(C_1^*)_{(1)}, \quad (D_1)_{(1)}, \quad (E_1)_{(1)}. \tag{18·14}$$

which were calculated on the ENIAC from the cards for the solution of the equations of order 0. It was convenient to carry out these calculations in such an order that the cards for each interval of the integration were punched in the deck-order 3, 1, (2, 1). A deck-number was punched on each card in the course of the calculation, and when finished they were sorted into separate decks by a card sorter. The sorter was then used to rearrange them in order of $\eta_{(1)}$, the cards for each single interval $\eta_{(0)}$ to $\eta_{(1)}$ being in deck-order 1, (2, 1), 3 for which they are wanted for the use of the data in formulae (18·1) to (18·8) (for the way of using the sorter to effect this rearrangement I am indebted to Miss K. McNulty). After the completion of the integration of these equations they can be again sorted into separate decks, and decks 1 and 3 similarly combined with deck (2, 2) for the equations for the second-order functions.

It seems practicable to use the ENIAC for the evaluation of the inhomogeneous terms C_n, D_n, E_n occurring in the equations for the functions of order higher than 1, and a scheme for the organization of this work has been drawn up, but so far it has not been tried.

The other aspect of the problem of the solution of the equations for the higher-order functions is the determination of the values of $h_n'(0)$ and $r_n(0)$ so that the solution should satisfy the conditions at infinity. The equations being linear, a formal possibility at least is to evaluate a particular integral and two complementary functions all satisfying the conditions

$$f_n(0) = h_n(0) = r_n'(0) = 0,$$

and form a linear combination of them so as to satisfy the conditions at infinity. For numerical work it is often found that this method is a purely formal possibility rather than a practically useful one, but in the present case it seems practicable, at least for $n = 1$ for which the complementary functions do not increase more rapidly than η^2 for large η, and it may be practicable for higher values of n.

The modification of the ENIAC set-up to evaluate the complementary functions is trivial; the accumulators which normally receive C_n, D_n and E_n at each interval from the constant transmitter are set to do nothing at that stage, instead of to receive, and to emit a programme pulse at the end of it as usual, so that the remainder of the calculation continues as before. For the complementary functions it is convenient to take the initial conditions

$$h_n'(0) = 1, \quad r_n(0) = 0,$$

and

$$h_n'(0) = 0, \quad r_n'(0) = 1,$$

or these multiplied by suitable powers of 10. Once the coefficients of the complementary functions have been determined, the formation of the linear combination of the particular integral and two complementary functions can be carried out on the ENIAC. This can be done very easily since the data required in this calculation are already on punched cards which is the form in which they would be required as input data to the machine.

If the trial values of $h_n'(0)$ and $r_n(0)$ are far from the right values, $h_n(0)$ or $r_n(0)$ or some intermediate quantity may build up to large values, and exceed the capacity of one of the accumulators. This would give spurious results, and might not be easy to detect if only one particular integral were evaluated. For this reason, the following procedure has been used for $n = 1$. Solutions for values of $h_1'(0)$, $r_1(0)$ forming a lattice in the $(h_1'(0), r_1(0))$ plane were first carried as far as $\eta = 3$, and for each one the value of $h_1'(3)$, $r_1'(3)$ recorded. The results were plotted in the $(h_1'(3), r_1'(3))$ plane; on account of the linearity of the equations, the curves $h_1'(0) = \text{const.}$ and $r_1(0) = \text{const.}$ in this plane should be strictly parallel straight lines, equally spaced if the values of $h_1'(0)$ and $r_1(0)$ are equally spaced. This, first, enables spurious solutions to be identified at a glance, without any detailed examination or analysis, and secondly, enables a good estimate to be made of the values of $h_1'(0)$ and $r_1(0)$ required to give $h_1'(3) = r_1'(3) = 0$ (or other values if any useful estimates can be made). The values of $h_1'(3)$ and $r_1'(3)$ will not be exactly 0 for the final solution, but they will probably be small, and a solution with initial values which makes them zero can be continued to $\eta = 4$ or 5 and the process repeated. When fairly good values of $h_1'(0)$ and $r_1(0)$ have been obtained by this procedure, the solution can be completed by evaluating a particular integral and two complementary functions and forming a linear combination.

19. Results

The results so far evaluated are the null-order functions for $\mathscr{M}_1^2 = 0, 2, 4, 6, 8, 10, 12, 16, 20$ and an approximation to the first-order functions for $\mathscr{M}_1^2 = 10$. The ENIAC was required for other work of higher priority before the exploratory work for $\mathscr{M}_1^2 = 10$, $n = 1$ had been completed, but it is hoped that the machine may be used to complete the solutions for the above values of $\mathscr{M}_1^2$ for $n = 1$ and to go to $n = 6$ or 7 for $\mathscr{M}_1^2 = 10$.

For $\mathscr{M}_1^2 = 20$, results were calculated with two values of the integration interval, $\delta\eta = 0{\cdot}02$ and $\delta\eta = 0{\cdot}04$, to provide an indication of the magnitude of the second-order integration errors, and to enable approximate corrections for them to be made by Richardson's (1927) 'h^2-extrapolation' process. If Δ represents the difference, at constant η, between the values of a quantity z calculated with $\delta\eta = 0{\cdot}02$ and $\delta\eta = 0{\cdot}04$, with the sign given by

$$\Delta z = (z \text{ calculated with } \delta\eta = 0{\cdot}04) - (z \text{ calculated with } \delta\eta = 0{\cdot}02), \tag{19·1}$$

the 'corrected' value of z given by this process is

$$(z \text{ corrected}) = (z \text{ calculated with } \delta\eta = 0{\cdot}02) - \tfrac{1}{3}\Delta z. \tag{19·2}$$

Some examples are given in table 2. It should be noted that the operations Δ, defined by (19·1), and $(\partial/\partial\eta)$ are not commutative, since part of the difference Δz in a quantity such as h_0, evaluated by integrating by either of the approximate formulae (17·4) or (17·5), arises from the aggregate second-order error in this integration formula. It can be seen on inspection of the results in table 2, or the fuller results in table 3, that $\int_0 \Delta h_0' \, d\eta$ differs considerably from Δh_0.

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Table 2. Example of h^2-extrapolation for h_0' and h_0. $\mathscr{M}_1^2 = 20$

η	0	0·6	1·2	1·8
$2h_0'$ $\delta\eta = 0{\cdot}04$	2·4871959	2·3941488	1·8362254	0·9085323
$2h_0'$ $\delta\eta = 0{\cdot}02$	2·4871024	2·3942014	1·8362670	0·9084410
$\Delta(2h_0')$	935	−526	−416	+913
$-\frac{1}{3}\Delta(2h_0')$	−312	+175	+139	−304
$2h_0'$ corrected	2·4870712	2·3942189	1·8362809	0·9084106
$\int_0 (h_0'$ corrected$)\, d\eta$	0	0·73910208	1·38802195	1·80129808
h_0 $\delta\eta = 0{\cdot}04$	0	0·73916955	1·38818596	1·80148906
h_0 $\delta\eta = 0{\cdot}02$	0	0·73911892	1·38806291	1·80134574
Δh_0	0	+5063	+12305	+14332
$-\frac{1}{3}\Delta h_0$	0	−1688	− 4102	− 4777
h_0 corrected	0	0·73910204	1·38802189	1·80129797
$(h_0$ corrected$) - \int_0 (h_0'$ corrected$)\, d\eta$	0	0·00000004	0·00000006	0·00000011

η	2·4	3·0	3·6
$2h_0'$ $\delta\eta = 0{\cdot}04$	0·2471652	0·0338846	0·0022677
$2h_0'$ $\delta\eta = 0{\cdot}02$	0·2471481	0·0339220	0·0022792
$\Delta(2h_0')$	+171	−374	−115
$-\frac{1}{3}\Delta(2h_0')$	− 57	+125	+ 38
$2h_0'$ corrected	0·2471421	0·0339345	0·0022830
$\int_0 (h_0'$ corrected$)\, d\eta$	1·96202215	1·99607725	1·99979088
h_0 $\delta\eta = 0{\cdot}04$	1·96213189	1·99610597	1·99979352
h_0 $\delta\eta = 0{\cdot}02$	1·96204949	1·99608432	1·99979141
Δh_0	+8240	+2165	+211
$-\frac{1}{3}\Delta h_0$	−2747	− 722	− 70
h_0 corrected	1·96202202	1·99607710	1·99979071
$(h_0$ corrected$) - \int_0 (h_0'$ corrected$)\, d\eta$	0·00000013	0·00000015	0·00000017

As a check, 'corrected' values of h_0' have been evaluated at intervals $\delta\eta = 0{\cdot}02$, and integrated by a formula correct to third differences (contributions from higher-order differences of h_0' are negligible, with this integration interval), and the results compared with the 'corrected' values of h_0. Examples are given in table 2. The residual differences are less than 2×10^{-7} in h_0, which is about 1/200 of the maximum value of the correction $-\frac{1}{3}\Delta h_0$. Part of this residual difference arises from the accumulation of rounding-off errors in the various integrations. But its general trend suggests that there is a systematic contribution, probably from higher-order error terms in the integration formulae. Similar results were obtained for r_0 and f_0, and suggest that the errors in the final 'corrected' values are not more than 1 in the sixth decimal.

The quantities actually recorded in the course of a solution with $\delta\eta = 0{\cdot}02$ were $2h_0'$, h_0, $4r_0'$, $2r_0$, f_0'', f_0; those finally tabulated† are h_0', h_0, r_0', r_0, f_0', f_0, rounded off to 6 decimals.

The variations of h_0, r_0 and f_0 with η are nearly the same for different values of $\mathscr{M}_1^2$ (see figures 11, 12), and it seems likely, therefore, that the aggregate second-order errors of the integration method are nearly independent of $\mathscr{M}_1^2$. The final results tabulated in table 4

† The results given in the tables are only about a fifth of the total available. The original ENIAC solutions, with explanatory notes, have been filed at Teddington, in the custody of the Secretary, Aeronautical Research Council, where they are available for reference in connexion with work for which it would be an advantage to have results at a smaller tabular interval.

Table 3. $\mathscr{M}_1^2=20$. Solution with $\delta\eta=0{\cdot}02$ and correction for finite interval length

η	$2h_0'$ ($\delta\eta=0{\cdot}02$)	$-\frac{1}{3}\times 10^7\Delta(2h_0')$	h_0' (corrected)	h_0 ($\delta\eta=0{\cdot}02$)	$-\frac{1}{3}\times 10^7\Delta h_0$	h_0 (corrected)
0	2·4871024	−312	1·243536	0	0	0
0·1	2·4866560	−224	1·243317	0·1243499	− 18	0·124348
0·2	2·4835854	−137	1·241786	0·2486236	− 40	0·248620
0·3	2·4752829	− 51	1·237639	0·3726247	− 66	0·372618
0·4	2·4591972	+ 32	1·229600	0·4960273	− 97	0·496018
0·5	2·4329040	+109	1·216457	0·6183809	−131	0·618368
0·6	2·3942014	175	1·197109	0·7391189	−169	0·739102
0·7	2·3412266	226	1·170625	0·8575724	−209	0·857551
0·8	2·2725831	257	1·136304	0·9729905	−251	0·972965
0·9	2·1874686	265	1·093748	1·0845668	−293	1·084537
1·0	2·0857847	247	1·042905	1·1914718	−335	1·191438
1·1	1·9682174	204	0·984119	1·2928906	−374	1·292853
1·2	1·8362670	139	0·918140	1·3880629	−410	1·388022
1·3	1·6922194	56	0·846112	1·4763233	−441	1·476279
1·4	1·5390498	− 35	0·769523	1·5571386	−465	1·557092
1·5	1·3802625	−125	0·690125	1·6301382	−482	1·630090
1·6	1·2196753	−206	0·609827	1·6951360	−490	1·695087
1·7	1·0611713	−268	0·530572	1·7521392	−488	1·752090
1·8	0·9084410	−304	0·454205	1·8013457	−478	1·801298
1·9	0·7647438	−313	0·382356	1·8431282	−459	1·843082
2·0	0·6327158	−294	0·316343	1·8780073	−431	1·877964
2·1	0·5142418	−252	0·257108	1·9066171	−397	1·906577
2·2	0·4104037	−193	0·205192	1·9296661	−359	1·929630
2·3	0·3215031	−125	0·160745	1·9478969	−317	1·947865
2·4	0·2471481	−057	0·123571	1·9620495	−275	1·962022
2·5	0·1863883	+006	0·093194	1·9728295	−233	1·972806
2·6	0·1378731	+057	0·068939	1·9808844	−193	1·980865
2·7	0·1000155	094	0·050012	1·9867874	−156	1·986772
2·8	0·0711418	117	0·035577	1·9910296	−124	1·991017
2·9	0·0496144	126	0·024813	1·9940190	− 96	1·994009
3·0	0·0339220	125	0·016967	1·9960843	− 72	1·996077
3·1	0·0227362	115	0·011374	1·9974832	− 53	1·997478
3·2	0·0149382	100	0·007474	1·9984120	− 38	1·998408
3·3	0·0096207	83	0·004814	1·9990166	− 27	1·999014
3·4	0·0060734	67	0·003040	1·9994023	− 18	1·999400
3·5	0·0037581	52	0·001882	1·9996435	− 11	1·999642
3·6	0·0022792	39	0·001142	1·9997914	− 7	1·999791
3·7	0·0013549	28	0·000679	1·9998803	− 4	1·999880
3·8	0·0007893	19	0·000396	1·9999326	− 2	1·999932
3·9	0·0004507	13	0·000226	1·9999628	− 1	1·999963
4·0	0·0002522	8	0·000126	1·9999799		1·999980
4·1	0·0001382	5	0·000069	1·9999894		1·999989
4·2	0·0000742	3	0·000037	1·9999945		1·999994
4·3	0·0000390	2	0·000020	1·9999972		1·999997
4·4	0·0000200	1	0·000010	1·9999986		1·999999
4·5	0·0000101	0	0·000005	1·9999994		1·999999
4·6	0·0000049		0·000002	1·9999997		2·000000
4·7	0·0000023		0·000001	1·9999999		2·000000
4·8	0·0000010		0·000001	2·0000000		2·000000
4·9	0·0000005		0·000000	2·0000000		2·000000
5·0	0·0000002			2·0000000		2·000000
5·1	0·0000001					
5·2	0·0000000					
5·3						
5·4						
5·5						

TABLE 3 (*cont.*)

η	$4r_0'$ ($\delta\eta = 0{\cdot}02$)	$-\frac{1}{3}\times 10^7 \Delta(4r_0')$	r_0' (corrected)	$2r_0$ ($\delta\eta = 0{\cdot}02$)	$-\frac{1}{3}\times 10^7 \Delta(2r_0)$	r_0 (corrected)
0	0	0	0	0·3368323	− 56	0·168413
0·1	−0·0442187	+ 7	−0·011054	0·3357266	− 56	0·167861
0·2	−0·0883237	+ 5	−0·022081	0·3324121	− 55	0·166203
0·3	−0·1320259	− 7	−0·033007	0·3269006	− 53	0·163448
0·4	−0·1748492	− 27	−0·043713	0·3192234	− 50	0·159609
0·5	−0·2161444	− 56	−0·054037	0·3094400	− 45	0·154718
0·6	−0·2551137	− 90	−0·063781	0·2976461	− 40	0·148821
0·7	−0·2908515	−128	−0·072716	0·2839803	− 33	0·141989
0·8	−0·3223995	−165	−0·080604	0·2686280	− 24	0·134313
0·9	−0·3488156	−198	−0·087209	0·2518225	− 15	0·125911
1·0	−0·3692523	−222	−0·092319	0·2338422	− 4	0·116921
1·1	−0·3830366	−236	−0·095765	0·2150038	+ 7	0·107502
1·2	−0·3897423	−236	−0·097441	0·1956520	19	0·097827
1·3	−0·3892456	−222	−0·097317	0·1761450	32	0·088074
1·4	−0·3817540	−194	−0·095443	0·1568395	44	0·078422
1·5	−0·3678027	−156	−0·091955	0·1380732	55	0·069039
1·6	−0·3482198	−111	−0·087058	0·1201495	66	0·060078
1·7	−0·3240612	− 64	−0·081017	0·1033245	75	0·051666
1·8	−0·2965271	− 19	−0·074132	0·0877974	82	0·043903
1·9	−0·2668685	+ 19	−0·066717	0·0737057	87	0·036857
2·0	−0·2362980	+ 49	−0·059073	0·0611250	90	0·030567
2·1	−0·2059147	67	−0·051477	0·0500728	90	0·025041
2·2	−0·1766479	76	−0·044160	0·0405156	89	0·020262
2·3	−0·1492260	76	−0·037305	0·0323784	86	0·016193
2·4	−0·1241661	69	−0·031040	0·0255551	81	0·012782
2·5	−0·1017836	57	−0·025444	0·0199188	75	0·009963
2·6	−0·0822139	43	−0·020552	0·0153316	68	0·007669
2·7	−0·0654436	28	−0·016360	0·0116525	60	0·005829
2·8	−0·0513438	14	−0·012836	0·0087444	53	0·004375
2·9	−0·0397048	+ 1	−0·009926	0·0064787	45	0·003241
3·0	−0·0302657	− 9	−0·007567	0·0047387	38	0·002371
3·1	−0·0227420	− 16	−0·005686	0·0034215	32	0·001712
3·2	−0·0168455	− 20	−0·004212	0·0024385	26	0·001221
3·3	−0·0123005	− 22	−0·003076	0·0017153	21	0·000859
3·4	−0·0088542	− 23	−0·002214	0·0011909	16	0·000596
3·5	−0·0062829	− 22	−0·001571	0·0008159	12	0·000409
3·6	−0·0043950	− 20	−0·001099	0·0005517	9	0·000256
3·7	−0·0030307	− 18	−0·000758	0·0003681	7	0·000184
3·8	−0·0020602	− 15	−0·000515	0·0002423	5	0·000121
3·9	−0·0013806	− 12	−0·000345	0·0001574	3	0·000079
4·0	−0·0009120	− 10	−0·000228	0·0001009	2	0·000051
4·1	−0·0005939	− 8	−0·000149	0·0000638	2	0·000032
4·2	−0·0003813	− 6	−0·000095	0·0000398	1	0·000020
4·3	−0·0002413	− 4	−0·000060	0·0000245	1	0·000012
4·4	−0·0001505	− 3	−0·000038	0·0000149		0·000007
4·5	−0·0000926	− 2	−0·000023	0·0000089		0·000004
4·6	−0·0000561	− 2	−0·000014	0·0000053		0·000003
4·7	−0·0000336	− 1	−0·000008	0·0000031		0·000002
4·8	−0·0000198	− 1	−0·000005	0·0000018		0·000001
4·9	−0·0000115	− 1	−0·000003	0·0000010		0·000001
5·0	−0·0000066	0	−0·000002	0·0000006		
5·1	−0·0000037		−0·000001	0·0000003		
5·2	−0·0000020		−0·000001	0·0000002		
5·3	−0·0000011			0·0000001		
5·4	−0·0000005					
5·5	−0·0000002					

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 3 (*cont.*)

η	f_0' ($\delta\eta = 0{\cdot}02$)	$-\frac{1}{3}\times 10^7(\Delta f_0')$	f_0' (corrected)	f_0 ($\delta\eta = 0{\cdot}02$)	$-\frac{1}{3}\times 10^7 \Delta f_0$	f_0 (corrected)
0	0	0	0	0	0	0
0·1	0·1055897	− 14	0·105588	0·0052789	− 1	0·005279
0·2	0·2112939	− 31	0·211291	0·0211219	− 4	0·021121
0·3	0·3171285	− 52	0·317123	0·0475422	− 8	0·047541
0·4	0·4230044	− 77	0·422997	0·0845491	− 14	0·084548
0·5	0·5287312	−106	0·528721	0·1321380	− 22	0·132136
0·6	0·6340208	−138	0·634007	0·1902805	− 33	0·190277
0·7	0·7384950	−173	0·738478	0·2589145	− 45	0·258910
0·8	0·8416942	−210	0·841673	0·3379361	− 59	0·337930
0·9	0·9430911	−250	0·943066	0·4271920	− 75	0·427186
1·0	1·0421074	−291	1·042078	0·5264734	− 93	0·526464
1·1	1·1381338	−332	1·138101	0·6355118	−113	0·635500
1·2	1·2305525	−372	1·230515	0·7539775	−135	0·753964
1·3	1·3187613	−411	1·318720	0·8814791	−160	0·881463
1·4	1·4021976	−446	1·402153	1·0175673	−186	1·017549
1·5	1·4803612	−476	1·480314	1·1617392	−214	1·161718
1·6	1·5528331	−500	1·552783	1·3134459	−243	1·313422
1·7	1·6192924	−517	1·619241	1·4721013	−273	1·472074
1·8	1·6795274	−527	1·679475	1·6370926	−304	1·637062
1·9	1·7334428	−526	1·733390	1·8077917	−336	1·807758
2·0	1·7810624	−519	1·781011	1·9835669	−368	1·983530
2·1	1·8225286	−500	1·822479	2·1637948	−399	2·163755
2·2	1·8580967	−473	1·858049	2·3478719	−429	2·347829
2·3	1·8881249	−441	1·888081	2·5352256	−457	2·535180
2·4	1·9130589	−404	1·913015	2·7253236	−483	2·725275
2·5	1·9334114	−362	1·933375	2·9176816	−507	2·917631
2·6	1·9497366	−317	1·949705	3·1118689	−528	3·111816
2·7	1·9626038	−272	1·962577	3·3075113	−547	3·307457
2·8	1·9725702	−229	1·972547	3·5042910	−563	3·504235
2·9	1·9801590	−189	1·980140	3·7019445	−577	3·701887
3·0	1·9858420	−152	1·985827	3·9002580	−588	3·900199
3·1	1·9900302	−120	1·990018	4·0990621	−597	4·099002
3·2	1·9930693	− 93	1·993060	4·2982250	−605	4·298165
3·3	1·9952423	− 71	1·995235	4·4976465	−611	4·497585
3·4	1·9967738	− 54	1·996768	4·6972517	−615	4·697190
3·5	1·9978387	− 40	1·997835	4·8969854	−618	4·896924
3·6	1·9985692	− 28	1·998566	5·0968080	−621	5·096746
3·7	1·9990639	− 19	1·999062	5·2966912	−622	5·296629
3·8	1·9993947	− 13	1·999393	5·4966152	−623	5·496553
3·9	1·9996132	− 9	1·999612	5·6965663	−624	5·696504
4·0	1·9997556	− 5	1·999755	5·8965352	−625	5·896473
4·1	1·9998475	− 3	1·999847	6·0965156	−625	6·096453
4·2	1·9999059	− 2	1·999906	6·2965035	−625	6·296441
4·3	1·9999426	− 1	1·999943	6·4964960	−625	6·496433
4·4	1·9999654	0	1·999965	6·6964915	−625	6·696429
4·5	1·9999794		1·999979	6·8964888	−625	6·896426
4·6	1·9999878		1·999988	7·0964871	−625	7·096425
4·7	1·9999929		1·999993	7·2964862	−625	7·296424
4·8	1·9999958		1·999996	7·4964856	−625	7·496423
4·9	1·9999975		1·999998	7·6964852	−625	7·696423
5·0	1·9999986		1·999999	7·8964850	−625	7·896423
5·1	1·9999993		1·999999	8·0964849	−625	8·096422
5·2	1·9999996		2·000000	8·2964848	−625	8·296422
5·3	1·9999998		2·000000	8·4964847	−625	8·496422
5·4				8·6964847	−625	8·696422
5·5						

include, for each $\mathscr{M}_1^2$, the corrections calculated for $\mathscr{M}_1^2 = 20$ and tabulated in table 3; they were applied to the recorded value and the results rounded off to 6 decimals in h_0', h_0, r_0', r_0, f_0', f_0 as for $\mathscr{M}_1^2 = 20$. Six decimals are given in the final results, since some of these functions may be multiplied by considerable factors in the equation for the higher-order functions;

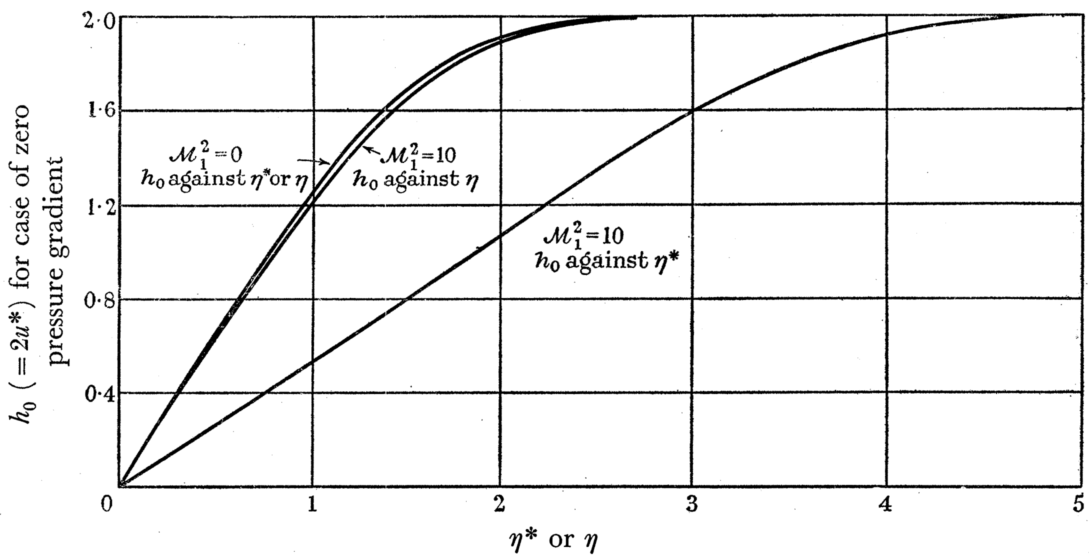

FIGURE 11. h_0 as a function of η^* and of η, for zero pressure gradient.

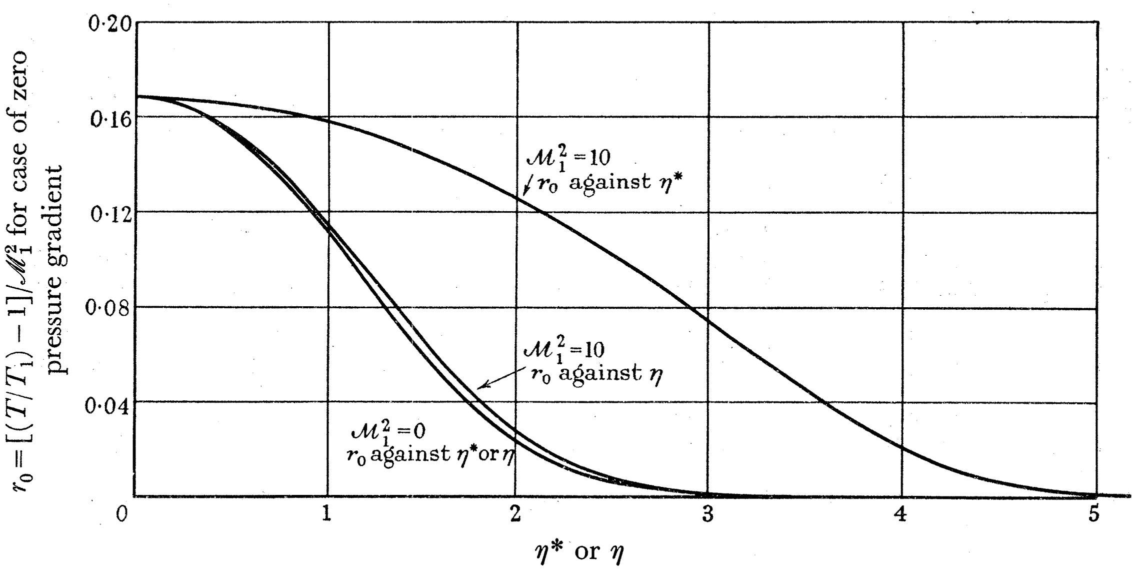

FIGURE 12. r_0 as function of η^* and of η, for zero pressure gradient.

even in the equations for the first-order functions, $3f_1$, which multiplies h_0' and r_0' (see equations (8·25 ii) and (8·27 ii)) is about 20 at $\eta = 3$. The quantity r_0 is tabulated to the same number of decimals as h_0 since, although it occurs multiplied by $\mathscr{M}_1^2$, which may be as large as 20, when this happens it is either in the form $(1+\mathscr{M}_1^2 r_0)^{1-\beta}$ or multiplied by $(1-\beta)$, and $(1-\beta) = \frac{1}{9}$ for the solutions evaluated.

The quantity $$\eta^* = \tfrac{1}{2}(U_1/\nu_1 x)^{\frac{1}{2}} y = \int_0 (\mu/\mu_1)\, d\eta$$

TABLE 4. NULL-ORDER FUNCTIONS

$\mathscr{M}_1^2 = 0$

η	h_0'	$h_0 = f_0'$	r_0'	r_0	f_0
0·0	1·328232	0·000000	0·000000	0·168945	0·000000
0·1	1·327936	0·132816	−0·012611	168314	0·006641
0·2	1·325878	265529	— 25181	166424	0·026560
0·3	1·320314	397876	— 37608	163283	0·059735
0·4	1·309553	529420	— 49721	158913	0·106108
0·5	1·292024	659563	— 61291	153356	165571
0·6	1·266351	787556	— 72041	146681	237948
0·7	1·231456	0·912528	— 81666	138985	322981
0·8	1·186649	1·033519	— 89853	130396	420320
0·9	1·131721	1·149522	−0·096312	121072	529517
1·0	1·067005	1·259538	−0·100808	111198	650023
1·1	0·993405	1·362628	— 103187	0·100980	781192
1·2	912371	1·457971	— 103395	0·090633	0·922289
1·3	825825	1·544917	−0·101494	80371	1·072505
1·4	736034	1·623026	−0·097657	70398	1·230975
1·5	645450	1·692095	— 92156	60895	1·396805
1·6	556521	1·752168	— 85332	52011	1·569091
1·7	471512	1·803526	— 77570	43860	1·746946
1·8	392350	1·846662	— 69262	36516	1·929521
1·9	320507	1·882237	— 60777	30014	2·116025
2·0	256937	1·911036	— 52438	24356	2·305740
2·1	202078	1·933913	— 44507	19513	2·498033
2·2	155887	1·951739	— 37176	15435	2·692353
2·3	0·117931	1·965364	— 30572	0·012054	2·888239
2·4	0·087481	1·975576	— 24760	0·009294	3·085311
2·5	0·063623	1·983080	— 19754	7075	3·283263
2·6	45364	1·988488	— 15529	5318	3·481856
2·7	31709	1·992308	−0·012030	3946	3·680906
2·8	21726	1·994953	−0·009185	2890	3·880277
2·9	0·014593	1·996749	— 6912	2089	4·079867
3·0	0·009608	1·997944	— 5127	1491	4·279605
3·1	6201	1·998724	— 3749	0·001050	4·479441
3·2	3923	1·999222	— 2702	0·000730	4·679340
3·3	2433	1·999535	— 1920	501	4·879279
3·4	0·001479	1·999727	−0·001345	339	5·079242
3·5	0·000881	1·999843	−0·000929	227	5·279221
3·6	515	1·999911	— 632	0·000149	5·479209
3·7	295	1·999951	— 424	0·000097	5·679202
3·8	0·000165	1·999973	— 281	62	5·879198
3·9	0·000091	1·999986	— 183	39	6·079196
4·0	49	1·999993	−0·000118	25	6·279195
4·1	26	1·999996	−0·000075	0·000015	6·479194
4·2	0·000013	1·999998	— 47	0·000009	6·679194
4·3	0·000007	1·999999	— 29	5	6·879194
4·4	4	2·000000	— 17	3	7·079194
4·5	2	2·000000	−0·000010	2	7·279194
4·6	0·000001	2·000000	−0·000006	0·000001	7·479194
4·7	0·000000	2·000000	— 3	0·000000	7·679194
4·8		2·000000	— 2		7·879194
4·9		2·000000	−0·000001		8·079194
5·0		2·000000	0·000000		8·279194

TABLE 4 (*cont.*)

$\mathscr{M}_1^2 = 2$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·312581	0·000000	0·000000	0·168824	0·000000	0·000000
0·1	1·312302	0·131251	−0·012316	168213	0·127090	0·006354
0·2	1·310357	262405	− 24594	166367	254164	0·025417
0·3	1·305093	393213	− 36737	163299	381058	0·057181
0·4	1·294907	523261	− 48587	159030	507450	0·101612
0·5	1·278299	651981	− 59928	153598	632863	158638
0·6	1·253945	778664	− 70500	147069	756675	228131
0·7	1·220786	0·902479	− 80013	139534	878135	309894
0·8	1·178116	1·022506	− 88167	131112	0·996391	403650
0·9	1·125670	1·137778	− 94683	121954	1·110521	509033
1·0	1·063685	1·247323	−0·099327	112237	1·219574	625583
1·1	0·992936	1·350223	−0·101938	0·102156	1·322622	752745
1·2	914719	1·445661	− 102448	0·091919	1·418809	0·889876
1·3	830798	1·532976	−0·100898	81735	1·507405	1·036252
1·4	743291	1·611700	−0·097434	71803	1·587850	1·191083
1·5	654532	1·681590	− 92299	62304	1·659792	1·353536
1·6	566887	1·742641	− 85812	53389	1·723107	1·522751
1·7	482593	1·795077	− 78335	45175	1·777905	1·697871
1·8	403596	1·839333	− 70248	37742	1·824517	1·878058
1·9	331430	1·876021	− 61915	31134	1·863467	2·062518
2·0	267143	1·905880	− 53658	25357	1·895426	2·250518
2·1	211284	1·929729	− 45745	20391	1·921169	2·441396
2·2	163925	1·948419	− 38382	16190	1·941523	2·634572
2·3	0·124737	1·962786	− 31705	0·012692	1·957316	2·829549
2·4	0·093077	1·973617	− 25792	0·009824	1·969344	3·025911
2·5	68099	1·981624	− 20669	7507	1·978334	3·223318
2·6	48848	1·987428	− 16320	5664	1·984931	3·421499
2·7	34351	1·991553	−0·012698	4219	1·989684	3·620244
2·8	23681	1·994427	−0·009738	3102	1·993047	3·819391
2·9	16003	1·996390	− 7360	2252	1·995385	4·018820
3·0	0·010601	1·997704	− 5484	1614	1·996982	4·218444
3·1	0·006884	1·998567	− 4028	0·001141	1·998054	4·418199
3·2	4382	1·999122	− 2916	0·000797	1·998763	4·618043
3·3	2734	1·999472	− 2081	549	1·999224	4·817944
3·4	1672	1·999688	− 1464	373	1·999519	5·017883
3·5	0·001003	1·999820	−0·001016	250	1·999706	5·217845
3·6	0·000589	1·999898	−0·000694	166	1·999822	5·417822
3·7	339	1·999943	− 468	0·000108	1·999894	5·617808
3·8	192	1·999969	− 311	0·000070	1·999937	5·817799
3·9	0·000106	1·999983	− 204	44	1·999963	6·017794
4·0	0·000057	1·999991	−0·000132	28	1·999979	6·217791
4·1	30	1·999996	−0·000084	17	1·999988	6·417790
4·2	0·000016	1·999998	− 53	0·000010	1·999993	6·617789
4·3	0·000008	1·999999	− 33	0·000006	1·999996	6·817788
4·4	4	1·999999	− 20	4	1·999998	7·017788
4·5	2	2·000000	−0·000012	2	1·999999	7·217788
4·6	0·000001	2·000000	−0·000007	1	1·999999	7·417788
4·7	0·000000	2·000000	− 4	0·000001	2·000000	7·617788
4·8		2·000000	− 2	0·000000	2·000000	7·817788
4·9		2·000000	−0·000001		2·000000	8·017788
5·0		2·000000	0·000000		2·000000	8·217788

Table 4 (*cont.*)

$\mathscr{M}_1^2 = 4$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·300051	0·000000	−0·000000	0·168743	0·000000	0·000000
0·1	1·299784	0·129998	−0·012082	168134	0·122777	0·006138
0·2	1·297922	259904	− 24128	166327	245586	0·024556
0·3	1·292885	389478	− 36046	163317	368318	55253
0·4	1·283132	518325	− 47686	159127	490721	0·098209
0·5	1·267220	645900	− 58843	153796	612394	0·153373
0·6	1·243866	771522	− 69270	147383	732801	220645
0·7	1·212031	0·894392	− 78687	139975	851283	299868
0·8	1·171002	1·013623	− 86807	131688	0·967072	390811
0·9	1·120478	1·128277	− 93358	122666	1·079326	493163
1·0	1·060629	1·237409	−0·098109	113076	1·187156	606527
1·1	0·992133	1·340115	−0·100894	0·103109	1·289674	730416
1·2	916173	1·435586	− 101637	0·092965	1·386033	0·864255
1·3	834389	1·523155	−0·100362	82849	1·475478	1·007391
1·4	748780	1·602335	−0·097194	72956	1·557392	1·159097
1·5	661575	1·672856	− 92357	63466	1·631331	1·318600
1·6	575072	1·734672	− 86148	54531	1·697056	1·485088
1·7	491471	1·787965	− 78913	46271	1·754545	1·657735
1·8	412720	1·833125	− 71018	38770	1·803992	1·835727
1·9	340393	1·870721	− 62821	32077	1·845792	2·018278
2·0	275608	1·901454	− 54644	26206	1·880503	2·204649
2·1	218996	1·926114	− 46760	21139	1·908813	2·394165
2·2	170724	1·945530	− 39380	16837	1·931484	2·586223
2·3	0·130546	1·960528	− 32651	13241	1·949308	2·780300
2·4	0·097897	1·971890	− 26662	0·010282	1·963069	2·975950
2·5	71986	1·980332	− 21446	0·007883	1·973502	3·172804
2·6	51899	1·986482	− 16997	5967	1·981272	3·370563
2·7	36683	1·990875	− 13274	4460	1·986958	3·568990
2·8	25419	1·993951	−0·010218	3290	1·991048	3·767902
2·9	17266	1·996063	−0·007752	2396	1·993941	3·967161
3·0	0·011497	1·997484	− 5797	1723	1·995954	4·166662
3·1	0·007504	1·998422	− 4273	0·001223	1·997334	4·366331
3·2	4801	1·999028	− 3105	0·000856	1·998265	4·566114
3·3	3011	1·999413	− 2225	592	1·998884	4·765974
3·4	1851	1·999652	− 1571	404	1·999290	4·965884
3·5	0·001116	1·999798	−0·001094	272	1·999554	5·165827
3·6	0·000659	1·999885	−0·000751	181	1·999723	5·365792
3·7	381	1·999936	− 508	0·000119	1·999829	5·565770
3·8	216	1·999965	− 339	0·000077	1·999896	5·765756
3·9	0·000120	1·999981	− 223	49	1·999937	5·965748
4·0	0·000066	1·999990	−0·000144	31	1·999962	6·165743
4·1	35	1·999995	−0·000092	19	1·999978	6·365740
4·2	0·000018	1·999997	− 58	0·000012	1·999987	6·565738
4·3	0·000009	1·999999	− 36	0·000007	1·999992	6·765737
4·4	5	1·999999	− 22	4	1·999996	6·965737
4·5	2	2·000000	−0·000013	2	1·999998	7·165736
4·6	0·000001	2·000000	−0·000008	1	1·999999	7·365736
4·7	0·000000	2·000000	− 5	0·000001	1·999999	7·565736
4·8			−0·000003	0·000000	2·000000	7·765736
4·9			−0·000000		2·000000	7·965736

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 4 (*cont.*)

$\mathscr{M}_1^2 = 6$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·289590	0·000000	−0·000000	0·168674	0·000000	0·000000
0·1	1·289333	0·128953	−0·011888	168079	0·119338	0·005966
0·2	1·287538	257816	− 23742	166297	238735	0·023870
0·3	1·282680	386359	− 35474	163335	358120	53713
0·4	1·273272	514201	− 46939	159211	477284	0·095486
0·5	1·257916	640816	− 57943	153962	595883	0·149151
0·6	1·235363	765545	− 68245	147646	713442	214628
0·7	1·204592	0·887615	− 77579	140345	829368	291784
0·8	1·164888	1·006167	− 85665	132171	0·942963	380423
0·9	1·115924	1·120285	− 92239	123262	1·053449	480273
1·0	1·057820	1·229047	− 97071	113782	1·160000	590981
1·1	0·991181	1·331565	−0·099994	0·103912	1·261771	712112
1·2	917099	1·427035	−0·100924	0·093849	1·357943	843147
1·3	837114	1·514787	− 99873	83793	1·447765	0·983488
1·4	753126	1·594323	− 96953	73937	1·530594	1·132465
1·5	667277	1·665349	− 92366	64458	1·605935	1·289355
1·6	581799	1·727790	− 86394	55510	1·673473	1·453390
1·7	498856	1·781792	− 79368	47214	1·733089	1·623783
1·8	420390	1·827708	− 71643	39659	1·784867	1·799745
1·9	348001	1·866071	− 63570	32897	1·829088	1·980504
2·0	282857	1·897549	− 55471	26946	1·866208	2·165325
2·1	225658	1·922906	− 47620	21795	1·896820	2·353528
2·2	176643	1·942952	− 40234	17407	1·921617	2·544496
2·3	135644	1·958501	− 33468	13728	1·941345	2·737683
2·4	0·102160	1·970331	− 27417	0·010690	1·956760	2·932621
2·5	0·075448	1·979159	− 22125	0·008219	1·968593	3·128916
2·6	54636	1·985618	− 17592	6239	1·977518	3·326244
2·7	38790	1·990252	− 13784	4676	1·984134	3·524344
2·8	27000	1·993512	−0·010645	3460	1·988958	3·723012
2·9	18422	1·995759	−0·008102	2528	1·992418	3·922091
3·0	0·012322	1·997279	− 6079	1823	1·994861	4·121462
3·1	0·008079	1·998286	− 4496	0·001297	1·996561	4·321039
3·2	5192	1·998940	− 3278	0·000911	1·997726	4·520757
3·3	3271	1·999357	− 2356	632	1·998514	4·720572
3·4	2020	1·999617	− 1669	432	1·999039	4·920451
3·5	0·001223	1·999777	−0·001166	292	1·999386	5·120374
3·6	0·000726	1·999872	−0·000803	194	1·999611	5·320324
3·7	422	1·999928	− 545	0·000128	1·999757	5·520293
3·8	241	1·999960	− 364	0·000083	1·999849	5·720274
3·9	0·000135	1·999979	− 240	53	1·999907	5·920262
4·0	0·000074	1·999989	− 156	33	1·999944	6·120255
4·1	39	1·999994	−0·000100	21	1·999966	6·320250
4·2	21	1·999997	−0·000063	0·000013	1·999980	6·520248
4·3	0·000011	1·999999	− 40	0·000008	1·999988	6·720246
4·4	0·000005	1·999999	− 24	4	1·999993	6·920245
4·5	3	2·000000	−0·000015	2	1·999996	7·120245
4·6	1	2·000000	−0·000009	1	1·999998	7·320244
4·7	0·000001	2·000000	− 5	0·000001	1·999999	7·520244
4·8	0·000000	2·000000	− 3	0·000000	1·999999	7·720244
4·9		2·000000	−0·000002		2·000000	7·920244

TABLE 4 (*cont.*)

$\mathscr{M}_1^2=8$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·280609	0·000000	−0·000000	0·168618	0·000000	0·000000
0·1	1·280357	0·128054	−0·011723	168032	0·116489	0·005824
0·2	1·278617	256022	− 23413	166274	233054	0·023300
0·3	1·273908	383680	− 34986	163353	349650	52436
0·4	1·264786	510657	− 46302	159286	466101	0·093226
0·5	1·249890	636445	− 57173	154107	582102	0·145641
0·6	1·228002	760403	− 67368	147873	697223	209616
0·7	1·198116	881779	− 76628	140664	810924	285038
0·8	1·159518	999736	− 84682	132587	0·922562	371732
0·9	1·111862	1·113382	− 91270	123777	1·031416	469457
1·0	1·055228	1·221810	− 96166	114390	1·136709	577895
1·1	0·990163	1·324146	−0·099202	0·104605	1·237643	696652
1·2	917686	1·419595	−0·100288	0·094614	1·333431	825251
1·3	839253	1·507484	− 99426	84613	1·423338	0·963141
1·4	756680	1·587307	− 96715	74791	1·506717	1·109700
1·5	672033	1·658750	− 92344	65325	1·583047	1·264248
1·6	587485	1·721715	− 86578	56368	1·651962	1·426061
1·7	505164	1·776320	− 79737	48045	1·713273	1·594386
1·8	426999	1·822885	− 72165	40446	1·766980	1·768461
1·9	354610	1·861911	− 64206	33625	1·813270	1·947534
2·0	289203	1·894039	− 56182	27607	1·852502	2·130879
2·1	231531	1·920009	− 48366	22382	1·885184	2·317815
2·2	181900	1·940613	− 40981	17919	1·911935	2·507718
2·3	140201	1·956653	− 34187	14166	1·933445	2·700028
2·4	0·105993	1·968903	− 28087	0·011059	1·950437	2·894257
2·5	0·078584	1·978078	− 22731	0·008524	1·963625	3·089989
2·6	57130	1·984818	− 18126	6488	1·973684	3·286878
2·7	40721	1·989673	− 14244	4875	1·981226	3·484642
2·8	28456	1·993101	−0·011031	3617	1·986788	3·683058
2·9	19494	1·995474	−0·008421	2649	1·990825	3·881950
3·0	0·013092	1·997084	− 6336	1915	1·993710	4·081185
3·1	0·008618	1·998157	− 4700	0·001366	1·995741	4·280664
3·2	5561	1·998856	− 3436	0·000962	1·997151	4·480313
3·3	3518	1·999303	− 2477	669	1·998116	4·680079
3·4	2181	1·999584	− 1760	459	1·998768	4·879926
3·5	0·001326	1·999756	−0·001233	311	1·999203	5·079826
3·6	0·000790	1·999860	−0·000851	207	1·999490	5·279762
3·7	461	1·999921	− 579	0·000137	1·999677	5·479721
3·8	264	1·999956	− 389	0·000089	1·999798	5·679695
3·9	0·000148	1·999976	− 257	57	1·999874	5·879679
4·0	0·000081	1·999987	− 168	36	1·999923	6·079669
4·1	44	1·999994	−0·000108	22	1·999954	6·279663
4·2	23	1·999997	−0·000068	0·000014	1·999972	6·479659
4·3	0·000012	1·999998	− 43	0·000008	1·999983	6·679657
4·4	0·000006	1·999999	− 26	5	1·999990	6·879655
4·5	3	2·000000	− 16	3	1·999994	7·079655
4·6	1	2·000000	−0·000010	2	1·999997	7·279654
4·7	0·000001	2·000000	−0·000006	0·000001	1·999998	7·479654
4·8	0·000000	2·000000	− 3	0·000000	1·999999	7·679654
4·9		2·000000	− 2		2·000000	7·879654
5·0			− 1			

TABLE 4 (*cont.*)

$\mathscr{M}_1^2 = 10$

η	h'_0	h_0	r'_0	r_0	f'_0	f_0
0·0	1·272733	0·000000	−0·000000	0·168571	0·000000	0·000000
0·1	1·272491	0·127267	−0·011579	167992	0·114064	0·005703
0·2	1·270798	254450	− 23127	166256	228217	0·022816
0·3	1·266214	381331	− 34561	163370	342431	51348
0·4	1·257334	507550	− 45747	159352	456553	0 091299
0·5	1·242830	632611	− 56502	154235	570310	0·142646
0·6	1·221507	755889	− 66602	148073	683308	205335
0·7	1·192377	876652	− 75795	140944	795047	279266
0·8	1·154726	0·994081	− 83818	132953	0·904930	364283
0·9	1·108192	1·107302	− 90415	124228	1·012282	460167
1·0	1·052825	1·215425	− 95363	114925	1·116370	566629
1·1	0·989124	1·317589	− 98494	0·105216	1·216439	683306
1·2	918045	1·413004	− 99712	0·095290	1·311735	809757
1·3	840974	1·500998	− 99013	85338	1·401544	0·945469
1·4	759656	1·581057	− 96484	75548	1·485226	1·089861
1·5	676088	1·652853	− 92301	66096	1·562249	1·242292
1·6	592388	1·716268	− 86718	57135	1·632215	1·402074
1·7	510651	1·771395	− 80042	48789	1·694888	1·568490
1·8	432796	1·818527	− 72609	41151	1·750201	1·740806
1·9	360449	1·858137	− 64756	34281	1·798264	1·918288
2·0	294847	1·890841	− 56803	28204	1·839355	2·100225
2·1	236791	1·917358	− 49024	22915	1·873900	2·285940
2·2	186635	1·938462	− 41644	18386	1·902445	2·474805
2·3	144332	1·954946	− 34830	14567	1·925624	2·666250
2·4	0·109489	1·967578	− 28689	0·011397	1·944117	2·859774
2·5	0·081459	1·977072	− 23279	0·008805	1·958615	3·054941
2·6	59429	1·984070	− 18612	6717	1·969785	3·251386
2·7	42512	1·989128	− 14663	5059	1·978246	3·448808
2·8	29814	1·992713	−0·011386	3762	1·984549	3·646964
2·9	20499	1·995204	−0·008714	2761	1·989170	3·845662
3·0	0·013816	1·996901	− 6574	2001	1·992506	4·044755
3·1	0·009129	1·998034	− 4889	1431	1·994878	4·244132
3·2	5912	1·998776	− 3584	0·001010	1·996541	4·443708
3·3	3754	1·999252	− 2590	0·000704	1·997691	4·643423
3·4	2336	1·999551	− 1845	484	1·998476	4·843234
3·5	0·001425	1·999736	−0·001296	328	1·999006	5·043110
3·6	0·000852	1·999848	−0·000897	220	1·999358	5·243029
3·7	499	1·999914	− 612	0·000145	1·999590	5·442978
3·8	287	1·999952	− 412	0·000095	1·999741	5·642945
3·9	0·000161	1·999974	− 273	61	1·999838	5·842924
4·0	0·000089	1·999986	− 179	39	1·999900	6·042911
4·1	48	1·999993	−0·000115	24	1·999939	6·242903
4·2	25	1·999996	−0·000073	0·000015	1·999963	6·442899
4·3	0·000013	1·999998	− 46	0·000009	1·999978	6·642896
4·4	0·000006	1·999999	− 28	5	1·999987	6·842894
4·5	3	2·000000	− 17	3	1·999993	7·042893
4·6	2	2·000000	−0·000010	2	1·999996	7·242892
4·7	0·000001	2·000000	−0·000006	0·000001	1·999998	7·442892
4·8	0·000000	2·000000	− 3	0·000000	1·999999	7·642892
4·9		2·000000	− 2		2·000000	7·842892
5·0		2·000000	− 1		2·000000	8·042892

TABLE 4 (*cont.*)

$\mathscr{M}_1^2 = 12$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·265725	0·000000	−0·000000	0·168531	0·000000	0·000000
0·1	1·265489	0·126567	−0·011452	167958	0·111959	0·005597
0·2	1·263837	253051	− 22874	166241	224015	0·022395
0·3	1·259362	379240	− 34185	163387	336154	50403
0·4	1·250692	504783	− 45255	159412	448243	0·089624
0·5	1·236527	629196	− 55907	154350	560029	0·140042
0·6	1·215695	751867	− 65922	148252	671151	201608
0·7	1·187222	872080	− 75055	141194	781140	274234
0·8	1·150395	0·989033	− 83048	133279	889437	357779
0·9	1·104842	1·101869	− 89650	124631	0·995406	452043
1·0	1·050586	1·209712	− 94642	115402	1·098355	556759
1·1	0·988086	1·311711	− 97853	0·105762	1·197563	671589
1·2	918244	1·407084	− 99186	0·095894	1·292309	796122
1·3	842386	1·495159	− 98629	85988	1·381903	0·929879
1·4	762193	1·575416	− 96260	76229	1·465716	1·072309
1·5	679603	1·647517	− 92245	66791	1·543216	1·222810
1·6	596686	1·711325	− 86826	57826	1·613987	1·380727
1·7	515501	1·766911	− 80299	49462	1·677759	1·545373
1·8	437955	1·814546	− 72993	41793	1·734416	1·716041
1·9	365678	1·854677	− 65239	34879	1·784007	1·892020
2·0	299932	1·887899	− 57353	28749	1·826737	2·072613
2·1	241554	1·914909	− 49612	23403	1·862960	2·257150
2·2	190949	1·936468	− 42241	18815	1·893155	2·445004
2·3	148116	1·953358	− 35411	14937	1·917895	2·635600
2·4	112709	1·966339	− 29237	0·011711	1·937815	2·828423
2·5	0·084120	1·976127	− 23780	0·009066	1·953577	3·023024
2·6	61568	1·983365	− 19057	6930	1·965834	3·219022
2·7	44185	1·988613	− 15050	5231	1·975203	3·416095
2·8	31090	1·992345	−0·011714	3898	1·982247	3·613985
2·9	21447	1·994946	−0·008987	2867	1·987458	3·812484
3·0	0·014504	1·996724	− 6796	2082	1·991253	4·011430
3·1	0·009615	1·997915	− 5066	1493	1·993976	4·210699
3·2	6248	1·998698	− 3723	0·001056	1·995901	4·410199
3·3	3980	1·999202	− 2697	0·000738	1·997244	4·609860
3·4	2485	1·999520	− 1926	508	1·998168	4·809634
3·5	0·001521	1·999717	−0·001356	346	1·998796	5·009484
3·6	0·000913	1·999836	−0·000941	232	1·999217	5·209386
3·7	537	1·999907	− 644	154	1·999497	5·409323
3·8	309	1·999948	− 434	0·000100	1·999680	5·609282
3·9	0·000175	1·999972	− 289	0·000063	1·999799	5·809256
4·0	0·000097	1·999985	− 189	41	1·999875	6·009240
4·1	52	1·999992	−0·000122	26	1·999924	6·209231
4·2	28	1·999996	−0·000078	16	1·999954	6·409225
4·3	0·000014	1·999998	− 49	0·000010	1·999972	6·609221
4·4	0·000007	1·999999	− 30	0·000006	1·999984	6·809219
4·5	4	2·000000	− 18	3	1·999991	7·009218
4·6	2	2·000000	−0·000011	2	1·999995	7·209217
4·7	0·000001		−0·000007	0·000001	1·999997	7·409216
4·8	0·000000		− 4	0·000000	1·999999	7·609216
4·9			− 2			
5·0			−0·000001			

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TABLE 4 (*cont.*)

$\mathscr{M}_1^2 = 16$

η	h_0'	h_0	r_0'	r_0	f_0'	f_0
0·0	1·253669	0·000000	−0·000000	0·168465	0·000000	0·000000
0·1	1·253442	0·125361	−0·011235	167903	0·108444	0·005422
0·2	1·251857	250643	− 22441	166219	216997	0·021693
0·3	1·247563	375642	− 33542	163418	325662	48825
0·4	1·239242	500022	− 44414	159518	434332	0·086825
0·5	1·225642	623315	− 54888	154549	542791	0·135684
0·6	1·205630	744936	− 64756	148561	650720	195366
0·7	1·178253	864195	− 73783	141626	757703	265797
0·8	1·142808	0·980318	− 81721	133840	863238	346858
0·9	1·098903	1·092475	− 88327	125326	0·966750	438376
1·0	1·046522	1·199816	− 93386	116227	1·067613	540119
1·1	0·986056	1·301509	− 96729	0·106706	1·165167	651788
1·2	918325	1·396784	− 98253	0·096941	1·258747	773020
1·3	844551	1·484972	− 97934	87117	1·247709	0·903384
1·4	766310	1·565545	− 95837	77414	1·431459	1·042388
1·5	685438	1·638146	− 92109	68004	1·509474	1·189484
1·6	603919	1·702611	− 86973	59039	1·581330	1·344077
1·7	523750	1·758975	− 80708	50647	1·646719	1·505534
1·8	446807	1·807468	− 73627	42925	1·705460	1·673198
1·9	374724	1·848498	− 66053	35938	1·757513	1·846402
2·0	308796	1·882618	− 58293	29720	1·802977	2·024481
2·1	249922	1·910492	− 50625	24276	1·842084	2·206785
2·2	198582	1·932854	− 43279	19584	1·875190	2·392697
2·3	154857	1·950463	− 36431	15604	1·902754	2·581638
2·4	0·118485	1·964070	− 30204	0·012278	1·925316	2·773081
2·5	0·088927	1·974387	− 24670	0·009540	1·943467	2·966555
2·6	65458	1·982059	− 19854	7320	1·957816	3·161648
2·7	47249	1·987654	− 15746	5545	1·968966	3·358012
2·8	33441	1·991656	−0·012308	4148	1·977484	3·555354
2·9	23204	1·994461	−0·009483	3063	1·983884	3·753438
3·0	15785	1·996390	− 7202	2233	1·988616	3·952075
3·1	0·010527	1·997690	− 5392	1607	1·992061	4·151118
3·2	0·006881	1·998549	− 3979	0·001142	1·994532	4·350455
3·3	4410	1·999106	− 2895	0·000801	1·996280	4·550001
3·4	2770	1·999459	− 2076	554	1·997499	4·749693
3·5	1705	1·999679	− 1468	378	1·998338	4·949488
3·6	0·001029	1·999813	−0·001023	255	1·998908	5·149352
3·7	0·000609	1·999893	−0·000703	170	1·999291	5·349263
3·8	353	1·999940	− 476	0·000111	1·999545	5·549206
3·9	201	1·999967	− 318	0·000072	1·999711	5·749169
4·0	0·000112	1·999982	− 209	46	1·999819	5·949146
4·1	0·000061	1·999991	−0·000136	29	1·999888	6·149132
4·2	33	1·999995	−0·000087	18	1·999931	6·349123
4·3	0·000017	1·999998	− 55	0·000011	1·999958	6·549117
4·4	0·000008	1·999999	− 34	0·000007	1·999975	6·749114
4·5	4	2·000000	− 21	4	1·999985	6·949112
4·6	2	2·000000	−0·000012	2	1·999991	7·149111
4·7	0·000001	2·000000	−0·000007	1	1·999995	7·349110
4·8	0·000000	2·000000	− 4	0·000001	1·999997	7·549110
4·9			− 2	0·000000	1·999998	7·749110
5·0			− 1		1·999999	7·949110

is given, for the case of zero pressure gradient and $\beta = \frac{8}{9}$ (and only then), by

$$\eta^* = \int_0 (1+\mathscr{M}_1^2 r)^{\frac{8}{9}}\, d\eta.$$

In figure 11, p. 55, $\frac{1}{2}h_0$, which for zero pressure gradient is the tangential velocity u^*, is shown as a function both of η and of η^* for $\mathscr{M}_1^2 = 10$, and also for $\mathscr{M}_1^2 = 0$ for which $\eta^* = \eta$. It will be seen how very much smaller is the variation of u^* with $\mathscr{M}_1^2$ when u^* is regarded as a function of η than when it is regarded as a function of η^*. Figure 12 shows similar results for r_0.

Table 5. Values of $2h_0'(0)$ and $2r_0(0)$

	final solution with $\delta\eta = 0{\cdot}02$		corrected for finite interval length			
$\mathscr{M}_1^2$	$2h_0'(0)$	$2r_0(0)$	$2h_0'(0)$		$2r_0(0)$	
0	2·656494	0·3378957	2·65646		0·337889	
				−3130		−232
2	2·625194	0·3376644	2·62516		0·337657	
				−2506		−173
4	2·600133	0·3374907	2·60010		0·337484	
				−2092		−138
6	2·579211	0·3373535	2·57918		0·337346	
				−1797		−112
8	2·561244	0·3372415	2·56121		0·337234	
				−1574		− 94
10	2·545498	0·3371470	2·54547		0·337140	
				−1402		− 81
12	2·531482	0·3370666	2·53145		0·337059	
				−2411		−130
16	2·507368	0·3369360	2·50734		0·336929	
				−2027		−104
20	2·487102	0·3368322	2·48707		0·336825	
	final solution with $\delta\eta = 0{\cdot}04$					
20	2·487195	0·3368499				

For $\mathscr{M}_1^2 = 0$, $h_0 = f_0'$, and f_0, f_0', f_0'' have been tabulated to 5 decimals by Howarth (1938); but this case was included in the present work in order to obtain r_0 and r_0', which were not evaluated by Howarth. The values of h_0 given in table 4 for $\mathscr{M}_1^2 = 0$ differ by up to 2 units in the fifth decimal from Howarth's values of f_0', and the value of $h_0'(0) = 1{\cdot}328232$ differs by 10 in the sixth decimal from the value $f_0''(0) = 1{\cdot}328242$ quoted by Howarth. A possible reason for this discrepancy is that the variation with $\mathscr{M}_1^2$ of the correction $-\frac{1}{3}\Delta h_0$ for the aggregate second-order integration error is larger than expected, but the variation required to account for it is quite considerable; $-\frac{1}{3}\Delta[2h_0'(0)]$ for $\mathscr{M}_1^2 = 20$ is -312×10^{-7}, whereas to make the solution for $\mathscr{M}_1^2 = 0$ evaluated with $\delta\eta = 0{\cdot}02$ agree with the result quoted by Howarth, a value of about -105×10^{-7} would be required. The general similarity of the behaviour of h_0 in the two cases makes such a large variation with $\mathscr{M}_1^2$ appear unlikely. The origin of the discrepancy has not been traced; the results are put on record since there is no reason from internal evidence to suspect them.

The values of $h_0'(0)$ and $r_0(0)$, for the solutions both recorded and 'corrected' as explained above, are given in table 5. They show considerable departures from a linear variation with $\mathscr{M}_1^2$, to a degree which was unexpected.

On the other hand, the functions h_1 and r_1, at constant η, vary considerably with $\mathscr{M}_1^2$. Results for $\mathscr{M}_1^2 = 0$ and 10 are given in table 6 and shown graphically in figures 13 and 14. The reason for this can be seen from an examination of the magnitudes of the terms in equation (8·24 ii) for f_1, from which it appears that if h_1 and r_1 varied only little with $\mathscr{M}_1^2$, the inhomogeneous term $\beta m h_0$ in the numerator in this formula, which term is proportional

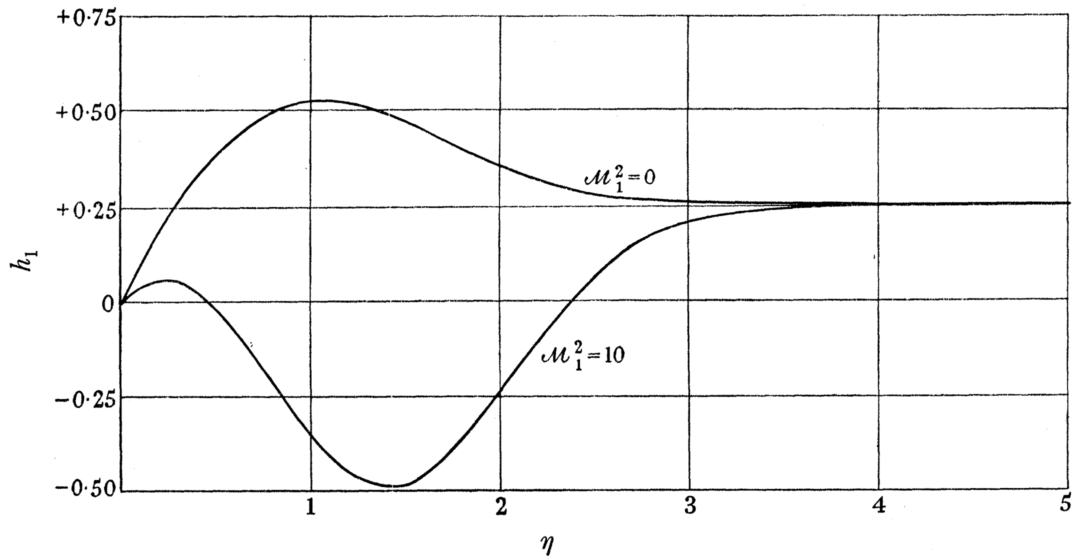

FIGURE 13. h_1 as a function of η.

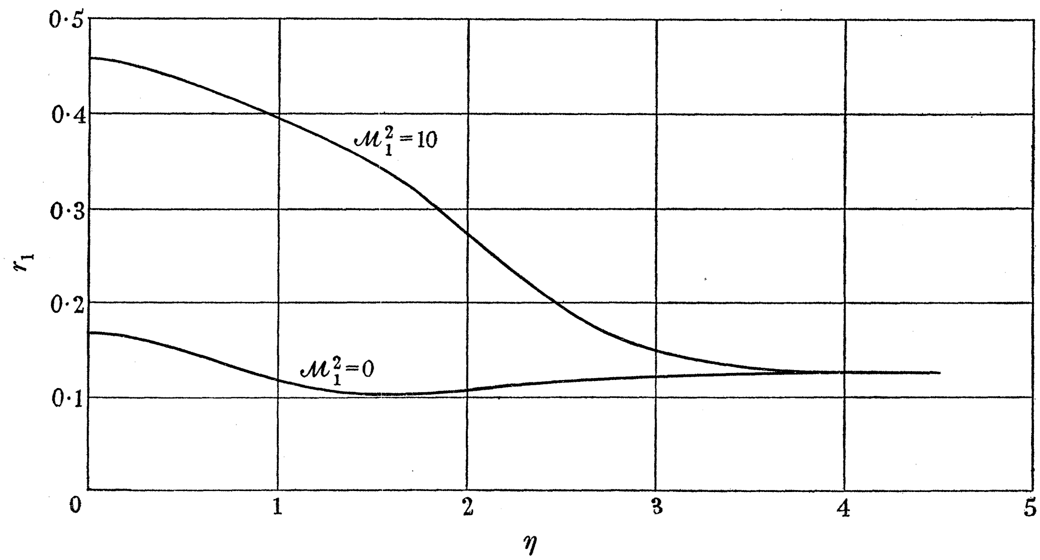

FIGURE 14. r_1 as a function of η.

to $\mathscr{M}_1^2$, would already be the predominant term for $\mathscr{M}_1^2 = 2$; moreover it is negative, so that already for $\mathscr{M}_1^2 = 2$, the sign of f_1 would be opposite to that for $\mathscr{M}_1^2 = 0$. It will be noticed that this term is absent if $\beta = 0$, that is, if the variation of viscosity with temperature is neglected. Its considerable effect on the solution shows how important it is to take account of the variation of the viscosity with temperature in considering flow with non-zero pressure gradient.

Table 6. First-order functions

$\mathscr{M}_1^2=0$

η	h_1'	$h_1=f_1'$	r_1'	r_1	$3f_1$
0·0	1·0205	0·0000	0·0000	0·1672	0·000
0·1	0·9205	0·0970	−0·0180	0·1663	0·015
0·2	0·8205	0·1841	−0·0333	0·1637	0·057
0·3	0·7206	0·2612	−0·0458	0·1597	0·124
0·4	0·6207	0·3282	−0·0555	0·1546	0·213
0·5	0·5212	0·3853	−0·0623	0·1487	0·320
0·6	0·4226	0·4325	−0·0660	0·1423	0·443
0·7	0·3255	0·4699	−0·0668	0·1356	0·579
0·8	0·2309	0·4977	−0·0648	0·1290	0·724
0·9	0·1404	0·5162	−0·0603	0·1227	0·876
1·0	0·0556	0·5260	−0·0537	0·1170	1·033
1·1	−0·0216	0·5276	−0·0452	0·1120	1·191
1·2	−0·0895	0·5220	−0·0356	0·1080	1·349
1·3	−0·1463	0·5101	−0·0256	0·1049	1·504
1·4	−0·1909	0·4931	−0·0156	0·1029	1·654
1·5	−0·2226	0·4723	−0·0062	0·1018	1·799
1·6	−0·2414	0·4490	+0·0021	0·1016	1·937
1·7	−0·2482	0·4244	0·0090	0·1021	2·068
1·8	−0·2442	0·3997	0·0143	0·1033	2·192
1·9	−0·2314	0·3759	0·0181	0·1050	2·308
2·0	−0·2120	0·3537	0·0203	0·1069	2·418
2·1	−0·1883	0·3336	0·0212	0·1090	2·521
2·2	−0·1624	0·3161	0·0211	0·1111	2·618
2·3	−0·1362	0·3012	0·0201	0·1132	2·711
2·4	−0·1113	0·2888	0·0185	0·1151	2·799
2·5	−0·0886	0·2788	0·0166	0·1169	2·884
2·6	−0·0688	0·2710	0·0145	0·1184	2·967
2·7	−0·0521	0·2650	0·0124	0·1198	3·047
2·8	−0·0385	0·2605	0·0104	0·1209	3·126
2·9	−0·0278	0·2572	0·0085	0·1218	3·203
3·0	−0·0196	0·2548	0·0068	0·1226	3·280
3·1	−0·0136	0·2531	0·0053	0·1232	3·356
3·2	−0·0091	0·2520	0·0041	0·1237	3·432
3·3	−0·0060	0·2513	0·0031	0·1240	3·507
3·4	−0·0039	0·2508	0·0023	0·1243	3·583
3·5	−0·0024	0·2505	0·0017	0·1245	3·658
3·6	−0·0015	0·2503	0·0012	0·1247	3·733
3·7	−0·0009	0·2502	0·0009	0·1248	3·808
3·8	−0·0005	0·2501	0·0006	0·1248	3·883
3·9	−0·0003	0·2501	0·0004	0·1249	3·958
4·0	−0·0002	0·2500	0·0003	0·1249	4·033
4·1	−0·0001	0·2500	0·0002	0·1250	4·108
4·2	−0·0001	0·2500	0·0001	0·1250	4·183
4·3	0·0000	0·2500	0·0001	0·1250	4·258
4·4			0·0000	0·1250	4·333
4·5				0·1250	4·408

Table 6 (*cont.*)

$\mathscr{M}_1^2=10$

η	h_1'	h_1	r_1'	r_1	$3f_1'$	$3f_1$
0·0	0·5198	0·0000	0·0000	0·4577	0·000	0·000
0·1	0·2806	0·0400	−0·0265	0·4563	−0·490	−0·024
0·2	0·0499	0·0564	−0·0469	0·4526	−1·043	−0·100
0·3	−0·1639	0·0506	−0·0613	0·4472	−1·657	−0·235
0·4	−0·3524	0·0245	−0·0704	0·4405	−2·325	−0·433
0·5	−0·5081	−0·0188	−0·0752	0·4332	−3·039	−0·701
0·6	−0·6243	−0·0758	−0·0767	0·4256	−3·788	−1·042
0·7	−0·6957	−0·1423	−0·0763	0·4180	−4·560	−1·459
0·8	−0·7190	−0·2134	−0·0753	0·4104	−5·340	−1·954
0·9	−0·6935	−0·2844	−0·0750	0·4029	−6·114	−2·527
1·0	−0·6214	−0·3505	−0·0766	0·3953	−6·864	−3·176
1·1	−0·5078	−0·4073	−0·0809	0·3874	−7·574	−3·899
1·2	−0·3610	−0·4510	−0·0882	0·3790	−8·230	−4·689
1·3	−0·1914	−0·4788	−0·0984	0·3697	−8·818	−5·542
1·4	−0·0113	−0·4890	−0·1108	0·3592	−9·326	−6·450
1·5	+0·1666	−0·4812	−0·1245	0·3475	−9·748	−7·405
1·6	+0·3302	−0·4561	−0·1382	0·3344	−10·079	−8·397
1·7	0·4693	−0·4159	−0·1505	0·3199	−10·321	−9·418
1·8	0·5763	−0·3633	−0·1603	0·3043	−10·477	−10·458
1·9	0·6472	−0·3018	−0·1667	0·2880	−10·556	−11·510
2·0	0·6813	−0·2351	−0·1692	0·2712	−10·569	−12·566
2·1	0·6812	−0·1667	−0·1676	0·2543	−10·528	−13·622
2·2	0·6520	−0·0999	−0·1621	0·2377	−10·447	−14·671
2·3	0·6004	−0·0371	−0·1533	0·2219	−10·340	−15·711
2·4	0·5328	+0·0197	−0·1418	0·2072	−10·220	−16·739
2·5	0·4593	0·0694	−0·1285	0·1936	−10·097	−17·755
2·6	0·3832	0·1115	−0·1140	0·1815	−9·977	−18·759
2·7	0·3105	0·1461	−0·0992	0·1708	−9·868	−19·751
2·8	0·2446	0·1738	−0·0847	0·1617	−9·773	−20·733
2·9	0·1875	0·1913	−0·0710	0·1539	−9·691	−21·706
3·0	0·1400	0·2116	−0·0585	0·1474	−9·624	−22·671
3·1	0·1019	0·2236	−0·0472	0·1422	−9·571	−23·631
3·2	0·0723	0·2323	−0·0371	0·1380	−9·529	−24·586
3·3	0·0501	0·2384	−0·0293	0·1346	−9·498	−25·537
3·4	0·0339	0·2425	−0·0225	0·1320	−9·474	−26·486
3·5	0·0223	0·2452	−0·0170	0·1300	−9·457	−27·432
3·6	0·0144	0·2470	−0·0126	0·1286	−9·444	−28·377
3·7	0·0091	0·2482	−0·0092	0·1275	−9·435	−29·321
3·8	0·0056	0·2490	−0·0066	0·1267	−9·429	−30·264
3·9	0·0034	0·2494	−0·0047	0·1261	−9·424	−31·207
4·0	0·0020	0·2497	−0·0032	0·1258	−9·422	−32·149
4·1	0·0012	0·2498	−0·0022	0·1255	−9·420	−33·091
4·2	0·0007	0·2499	−0·0015	0·1253	−9·418	−34·033
4·3	0·0004	0·2499	−0·0010	0·1252	−9·417	−34·975
4·4	0·0002	0·2500	−0·0007	0·1251	−9·417	−35·917
4·5	0·0001	0·2500	−0·0005	0·1251	−9·417	−36·858
4·6	0·0001	0·2500	−0·0003	0·1250	−9·417	−37·800
4·7	0·0000	0·2500	−0·0002	0·1250	−9·417	−38·742
4·8		0·2500	−0·0001	0·1250	−9·417	−39·684
4·9		0·2500	−0·0001	0·1250	−9·417	−40·625
5·0		0·2500		0·1250	−9·417	−41·567

20. The heat transfer case. (*Added in proof*)

As originally written, the present paper was restricted to the case of no heat transfer. But, because of the interest now being taken in certain cases of heat transfer at high gas speeds, it is desirable to indicate very briefly the changes in treatment needed to pass to the case when there is a heat transfer.

The essential difference is that the plate is maintained at a prescribed temperature which will be denoted by T_W (leaving T_w to denote the temperature, sometimes called the 'natural' temperature, which it attains when there is no heat transfer), which may be a (known) function of x. This implies that $(\partial T/\partial y)_w \neq 0$, and hence that $r_n'(0)$ is non-zero for at least some n, and is given.

The treatment used in this paper, applied to this case, indicates that the heat transfer coefficient should be related to $T_W - T_w$ and not to $T_W - T_1$. It can be argued that the same conclusion would apply to a turbulent layer, though the argument is suggestive rather than conclusive.

Previous work in this field seems to have been confined to the case of no pressure drop, constant T_W, and $\mathcal{M}_1 = 0$. Pohlhausen (F.D., p. 623) obtained a solution of equation (11·4i), with the dissipation term omitted. He showed that a very good approximation to the results obtained from the accurate solution was given by the formulae, in the usual notation,

$$\left.\begin{aligned} \mathcal{N} &= 0{\cdot}664\,\sigma^{\frac{1}{3}}\mathcal{R}^{\frac{1}{2}}, \\ \mathcal{S} &= 0{\cdot}664\,\sigma^{-\frac{2}{3}}\mathcal{R}^{-\frac{1}{2}}, \end{aligned}\right\} \qquad (20)$$

where both $\mathcal{N}$ and $\mathcal{S}$ are related to $T_W - T_1$. Eckert & Drewitz (1940), (quoted by Jacob 1947) showed that the inclusion of the dissipation term in the equation left these formulae unaffected if T_W were substituted for T_1, and gave reasons for thinking that this result was generally true.

The analysis of this case, based on the treatment of the present paper, confirms their conjecture both for any value of $\mathcal{M}_1$, and for functions of any order. Moreover both the calculations by Emmons & Brainerd (1941, 1942) and those reported in this paper show that C_f, as a function of $\mathcal{R}$, varies only slowly with $\mathcal{M}_1$; so it seems likely that the relations (20), properly interpreted, could be used to predict the heat transfer from a flat plate in a uniform supersonic stream up to $\mathcal{M}_1^2 = 20$ with ample accuracy for most practical purposes. In this connexion, it is worth recalling that the approximate formula

$$T_W = T_1[1 + \tfrac{1}{2}(\gamma - 1)\,\mathcal{M}_1^2\,\sigma^{\frac{1}{2}}]$$

is also of use up to these values of $\mathcal{M}_1$.

No attempt has been made to examine in detail the computing technique involved in the solution of the general equations of the heat transfer case. But it seems likely that, once the programme outlined in §19 has been completed, the methods outlined in §§12 and 13, with appropriate changes in boundary conditions, could be used in an iterative process without a prohibitive amount of labour.

Part of the work described above was carried out in the Engineering Division, National Physical Laboratory, on behalf of the Chief Scientific Officer, Ministry of Supply, by whose permission this paper is published. This work was carried out by us jointly (§§1 to 5 and §20 mainly by W.F.C. and a first version of §6 and §§8 to 12 by D.R.H.).

The remainder was carried out by one of us (D. R. H.) in the course of a visit to the Moore School of Electrical Engineering of the University of Pennsylvania, at the invitation of the Ordnance Department of the U.S. War Department, to study the ENIAC and its possible applications. He wishes to express his deep appreciation of the opportunity of making this visit and of making first-hand acquaintance of the ENIAC, and his thanks to Colonel P. N. Gillon, of the Office of the Chief of Ordnance, for making the arrangements for this visit, to Dr L. S. Dederick, of the Ballistics Research Laboratory, Aberdeen, Maryland, for agreeing to make the ENIAC available for the work considered in §§ 15 to 18, to Dr and Mrs H. H. Goldstine, to Dr D. H. Lehmer and other members of the group engaged in the operation of the machine, and especially to Miss K. McNulty, for instruction, advice and help in organizing the work for the machine, planning the machine set-up for it, and in running the machine. The active and friendly help received made the work, in addition to being of absorbing interest, a real pleasure. He also wishes to thank Miss Mumford and Mrs Pryor for assistance in carrying out the work involved in the numerical evaluation of various solutions by the iterative methods indicated in §§ 12 and 13.

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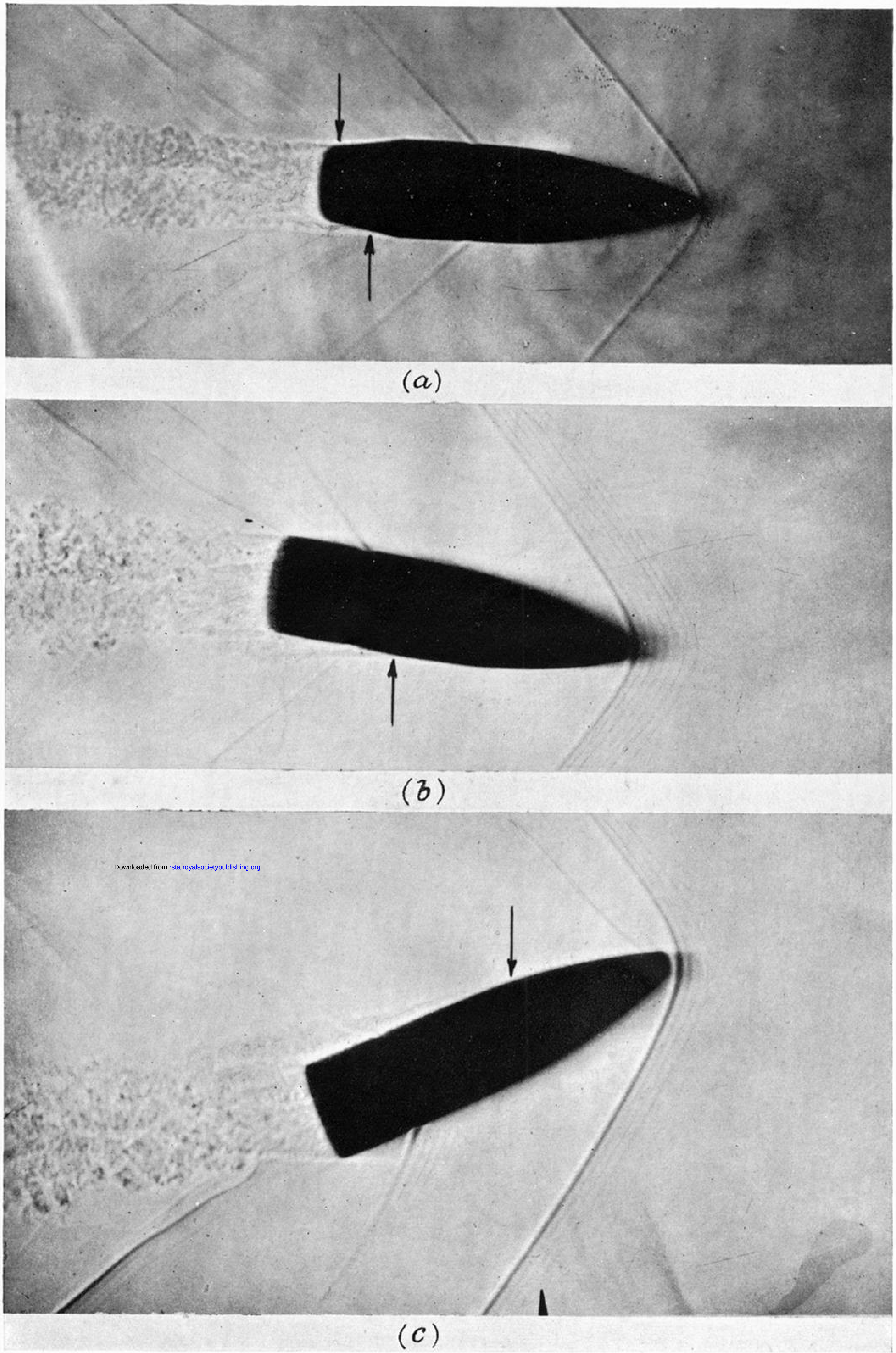

FIGURE 1. Photographs of 0·303 in. bullets in flight showing boundary layer separation. Nominal velocity 1700 ft. per sec. Mach number about $1\frac{1}{2}$. Arrows point to approximate position of separation. (*a*) Mark VIII z. Yaw small. (*b*) Mark VII. Yaw 10°. (*c*) Mark VII. Yaw 25°.

(*Facing p.* 4)